An Unconditionally Stable Cut Finite Element Immersed Boundary Method

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CRREL Ron Liston Seminar
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Outline

1. Motivation and finite element background
2. Introduction
   - The immersed boundary method
   - Cut finite element method (CutFEM)
3. Cut finite element immersed boundary method
   - Derivation of the algorithm
   - Energy stability

Motivation

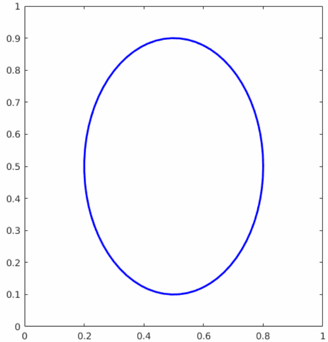


Figure: Immersed interface oscillating in fluid.

- **Goal:** Couple fluid velocity and pressure with forces generated by immersed elastic structure.
- Fluid initially at rest.
- Infinitesimally thin membrane.
  - Initially deformed.
  - Bending and stretching allowed.
- No external forces.

Finite Element Methods

- Finite element method: Numerical method that takes a larger problem and break down into smaller parts (elements).
- Necessary spaces of functions:
  - $L^2(\Omega)$: All functions f such that $\|f\|_{L^2(\Omega)}^2 = \int_\Omega f^2$ is finite.
  - $H^1(\Omega)$: All functions f such that $\|f\|_{H^1(\Omega)}^2 := \|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2$ is finite.
  - $H_0^1(\Omega)$: All functions $f \in H^1(\Omega)$ such that $f = 0$ on $\partial\Omega$.
- Recall: Green's identity (integration by parts)

$$-\int_\Omega (\Delta u)v = \int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} (\nabla u \cdot \hat{\mathbf{n}})v.$$

Weak Formulation

- Weak formulation: Multiply PDE by "sufficiently smooth" test function v and integrate by parts.
- Example: For $\Omega = [0,1] \times [0,1]$ and $f \in L^2(\Omega)$,

$$\begin{cases} -\Delta u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

  becomes

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \nabla u \cdot \hat{n} v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega).$$

- Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v, \qquad \forall v \in H_0^1(\Omega).$$

Finite Element Approximation

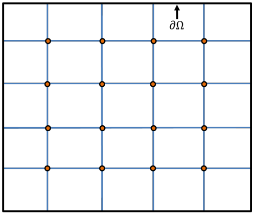
- Write approximate solution as sum of M scaled "hat" functions ϕ_i:

$$u_h = \sum_{i=1}^{M} c_i \phi_i.$$

- For $j = 1, \ldots, M$, choose $v = \phi_j$.
  - Yields M equations in M variables.
- Numerically solve

$$\sum_{i=1}^{M} c_i \int_{\Omega_h} \nabla\phi_i \cdot \nabla\phi_j = \sum_{i=1}^{M} \int_{\Omega_h} f\phi_j,$$

for $j = 1, \ldots, M$.



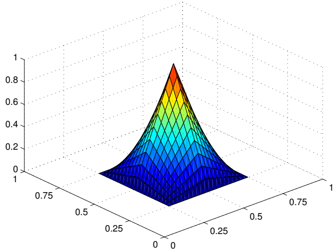


Figure: Example quadrilateral mesh and Q1 basis function.

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Original Problem Setting

- Solve incompressible Stokes equations on $\Omega = (0,1)^2$:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \mu \nabla \cdot \varepsilon(\mathbf{u}) + \nabla p &= \mathbf{F} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned}$$

  where $\varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$.

- $\Gamma(t)$: Elastic structure immersed in fluid domain Ω.
- $\mathbf{F}$: Forcing function defined later.

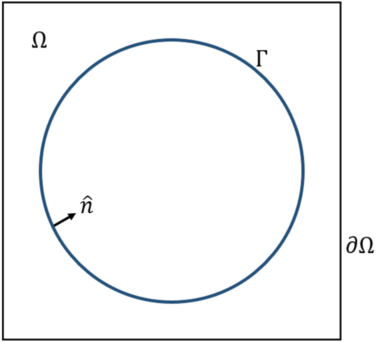

Figure: Example domain with notation used throughout talk.

Description of Γ

- Cartesian coordinates of points on $\Gamma(t)$ denoted by $\mathbf{X}(s, t)$ such that
  - s in reference interval $[0, L]$.
  - $\mathbf{X}(0, t) = \mathbf{X}(L, t)$.
- $\mathbf{X}(s, t)$ denotes position of a material point at time t.
- Parameter s considered a Lagrangian coordinate.
  - Generally *not* arc length.
  - Can create "stretching" of Γ with choice of parameterization.

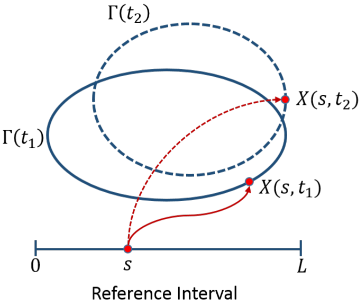

Figure: Mapping from reference interval to points on $\Gamma(t)$.

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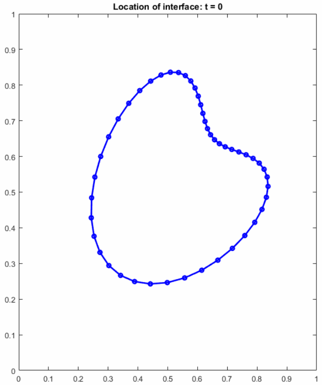

The Immersed Boundary Method

- Original immersed boundary method introduced by Charles Peskin (1972).
- Developed for numerical analysis of cardiac blood flow.
- General force defined by

$$\mathbf{F}(\mathbf{x}, t) = \int_0^L \mathbf{f}(s, t)\delta(\mathbf{x} - \mathbf{X}(s, t))\, ds.$$

- Simple model: Stressed initial configuration, zero length at rest

$$\mathbf{f}(s, t) = \kappa \frac{\partial^2 \mathbf{X}}{\partial s^2}(s, t).$$

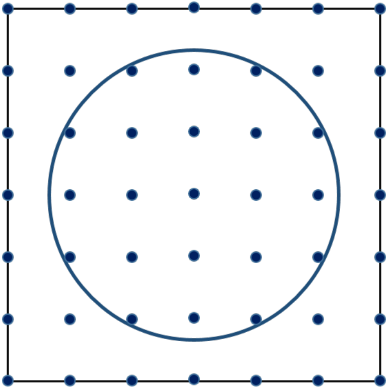

Figure: Immersed boundary in finite difference grid.

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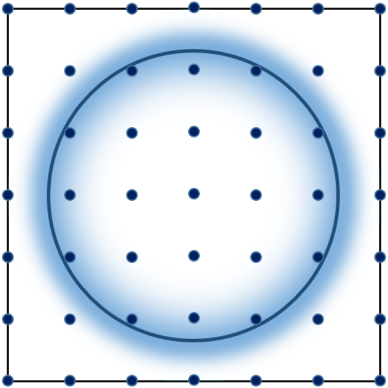


Figure: Dirac delta mollified to be seen by grid points.

Finite Element Immersed Boundary Method

- Finite element approach introduced by Boffi & Gastaldi.
- Incorporates force accurately by integrating over Dirac delta.
- **First-order accuracy obtained near interface.**
    - Error incurred from global approximation of discontinuous quantities and derivatives.

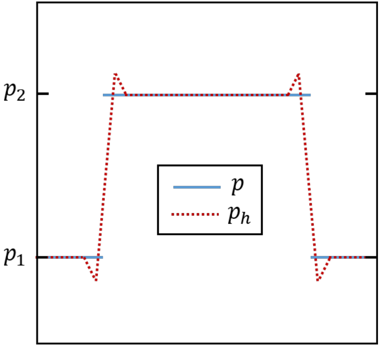


Figure: Dramatization of fitting global linear FEM approximation to discontinuous pressure.

One Potential Approach

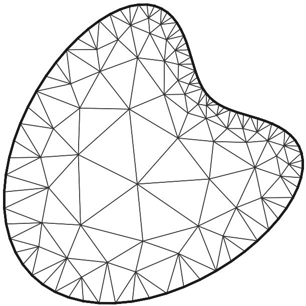


Figure: Triangular mesh fitted to interior subdomain.

- Capture jumps by meshing each subdomain.
  - Compute solution on interior and exterior separately.
- Meshing algorithms can be expensive.
- Subdomains change in time:
  - Allow mesh to deform, or
  - Re-mesh at each time step.

The Cut Finite Element Method (CutFEM)

- CutFEM partitions domains into Ω_1 (exterior) and Ω_2 (interior).
- Separate solution for Ω_1 and Ω_2 captures jumps.
  - Can prescribe jump of $\mathbf{u}$, p, and normal derivatives.
- **Optimal accuracy obtained near interface.**
- Integrate over intersection of each element with subdomain.
- Weakly impose boundary and jump conditions.

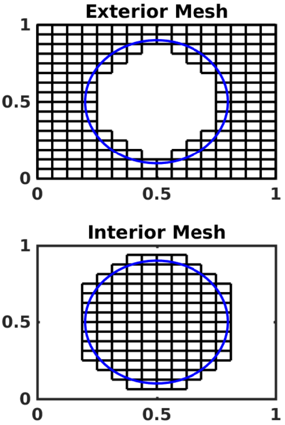
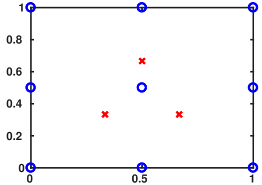
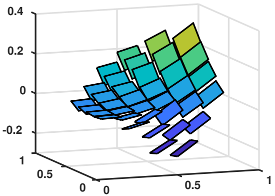


Figure: Example meshes $\Omega^e_{i,h}$ for CutFEM with interface.

Spatial Discretization



(a) DOFs: Blue ○ for velocity, red × for pressure



(b) Discontinuous pressure

- Uniform square mesh on Ω.
  - h: side length of each square.
- Velocity space $\mathbf{V}_{i,h}$: Piecewise biquadratic (continuous Q2) on $\Omega^e_{i,h}$.
- Pressure space $M_{i,h}$: Piecewise linear (discontinuous P1) on $\Omega^e_{i,h}$.
- For ϕ_i defined on Ω_i, denote jump across Γ by

$$[\![\phi]\!] = \phi_1 - \phi_2$$

and average by

$$\{\phi\} = \frac{1}{2}(\phi_1 + \phi_2).$$

Nitsche's Formulation for Laplace's Equation

- Strong form:

$$
\begin{aligned}
-\Delta u_i &= 0 \quad && \text{in } \Omega_i, \\
[\![\nabla u \cdot \hat{\mathbf{n}}]\!] &= g \quad && \text{on } \Gamma, \\
[\![u]\!] &= 0 \quad && \text{on } \Gamma, \\
u_1 &= 0 \quad && \text{on } \partial\Omega.
\end{aligned}
$$

- Multiply by test function and integrate by parts on Ω_1 and Ω_2 separately.
- After algebra, u_1 and u_2 coupled on Γ with jumps and averages. $\Longrightarrow$

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- Multiply by test function and integrate by parts on Ω_1 and Ω_2 separately.
- After algebra, u_1 and u_2 coupled on Γ with jumps and averages. $\implies$

- Add penalty terms with $\gamma > 0$ to enforce Dirichlet BCs (red).
- Note: Can add terms for symmetry.
- Weak form:

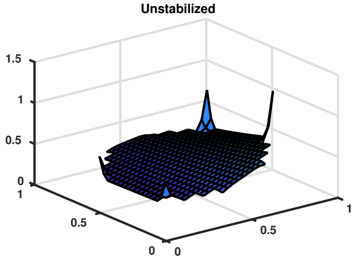
$$\sum_{i=1}^{2} (\nabla u_i, \nabla v_i)_{\Omega_i} - ([\![\nabla u \cdot \hat{\mathbf{n}}]\!], \{v\})_\Gamma - (\{\nabla u \cdot \hat{\mathbf{n}}\}, [\![v]\!])_\Gamma - (\nabla u_1 \cdot \hat{\mathbf{n}}_1, v_1)_{\partial\Omega} + \frac{\gamma}{h}([\![u]\!], [\![v]\!])_\Gamma + \frac{\gamma}{h}(u_1, v_1)_{\partial\Omega} = 0.$$

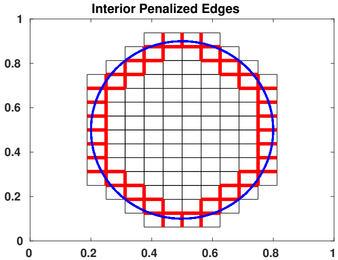
CutFEM: Ghost Penalty

- Add stabilizing "ghost penalty" terms to finite element problem:

$$\mathbf{j}_{i,h}(\mathbf{u}_i, \mathbf{v}_i) = \sum_{\ell=0}^{1} \sum_{F \in \mathcal{F}_{i,h}^{\Gamma}} \int_F h^{2\ell+1} \left[\partial_{\hat{\mathbf{n}}_F}^{(\ell)} (\varepsilon(\mathbf{u}_i)\hat{\mathbf{n}}_F)\right] \cdot \left[\partial_{\hat{\mathbf{n}}_F}^{(\ell)} (\varepsilon(\mathbf{v}_i)\hat{\mathbf{n}}_F)\right],$$

$$J_{i,h}(p_i, q_i) = \sum_{\ell=0}^{1} \sum_{F \in \mathcal{F}_{i,h}^{\Gamma}} \int_F h^{2\ell+1} \left[\partial_{\hat{\mathbf{n}}_F}^{(\ell)} p_i\right] \left[\partial_{\hat{\mathbf{n}}_F}^{(\ell)} q_i\right].$$





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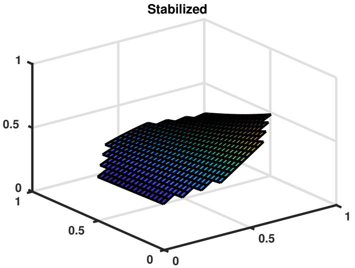

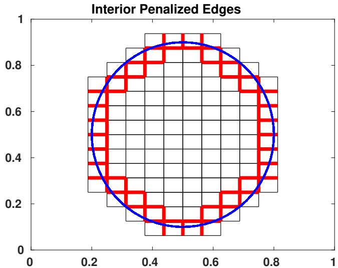

Steady-State Convergence

	Velocity				Pressure			
$1/h$	L^2	Rate	H^1	Rate	L^2	Rate	H^1	Rate
8	1.6862e-2		2.1210e-1		1.0151e-1		1.7770e-0	
16	3.0799e-4	54.8	1.1230e-2	18.9	1.2345e-2	8.2	7.2290e-1	2.5
32	1.6206e-5	19.0	1.2870e-3	8.7	2.0347e-3	6.1	3.0689e-1	2.4
64	1.0693e-6	15.2	2.0910e-4	6.2	4.7168e-4	4.3	1.4578e-1	2.1
128	7.6314e-8	14.0	3.2012e-5	6.5	2.0651e-4	2.3	7.0759e-2	2.1

Table: Error tables for the velocity and pressure..

- Above table computed solving steady-state Stokes equations with immersed interface.
  - Prescribed boundary conditions match actual solution.
- Observe optimal spatial convergence rate.
  - Velocity: 8 in L^2 norm, 4 in H^1 norm.
  - Velocity: 4 in L^2 norm, 2 in H^1 norm.

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The Problem

- Solve incompressible Stokes equations on Ω_1 and Ω_2:

$$
\begin{aligned}
\frac{\partial \mathbf{u}_i}{\partial t} - \mu \nabla \cdot \varepsilon(\mathbf{u}_i) + \nabla p_i = \mathbf{0} \quad & \text{in } \Omega_i(t), \\
\nabla \cdot \mathbf{u}_i = 0 \quad & \text{in } \Omega_i(t),
\end{aligned}
$$

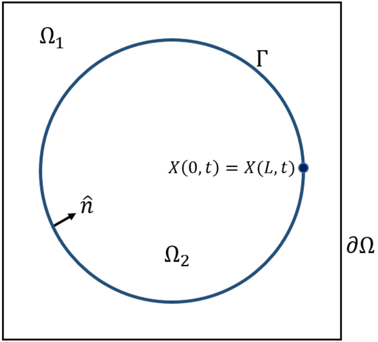

Figure: Problem separated into two domains.

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[\![(\mu\varepsilon(\mathbf{u}) - p)\hat{\mathbf{n}}]\!] = \frac{\kappa}{\left|\frac{\partial \mathbf{X}}{\partial s}\right|}\frac{\partial^2 \mathbf{X}}{\partial s^2} \quad & \text{on } \Gamma(t), \\
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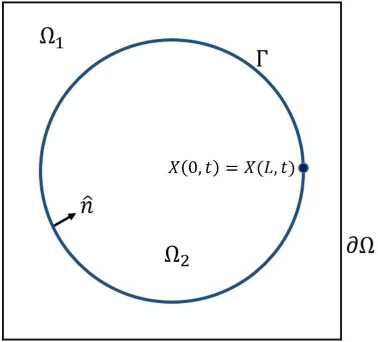
$$



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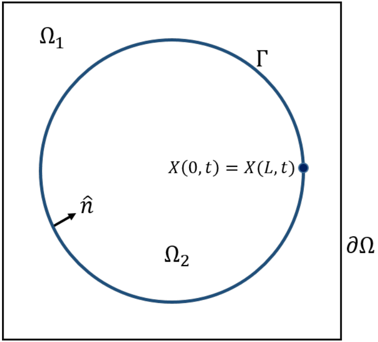

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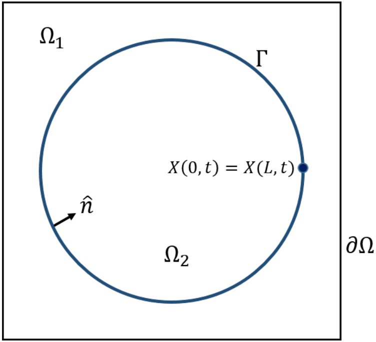
$$

- Interface no-slip condition:

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \{\mathbf{u}(\mathbf{X}(s,t),t)\}.$$



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Discretization of Domain

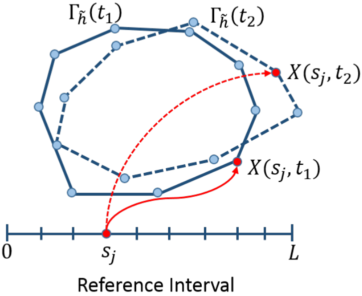


Figure: Each vertex of polygon tracked and mapped to $[0, L]$.

- $\Gamma_{\tilde{h}}$: Polygonal approximation to interface.
  - Sample $\mathbf{X}(s)$ at $m+1$ points.
  - Evenly-spaced points in $[0, L]$ denoted by $s_j = j\tilde{h}$ for $\tilde{h} = L/m$ and $j = 0, 1, \ldots, m$.
- Location of $\mathbf{X}(s_j)$ updated using fluid velocity at each time step.
- Creates polygonal approximations to subdomains $\Omega_{i,\tilde{h}}$.

Temporal Discretization

- For $\Delta t > 0$, discretize time derivative by

$$\frac{\partial \mathbf{u}_i^{n+1}}{\partial t} \approx \frac{\mathbf{u}_i^{n+1} - \mathbf{u}_i^n}{\Delta t}.$$

- Discrete no-slip condition:

$$\mathbf{X}_j^{n+1} = \mathbf{X}_j^n + \Delta t \left\{ \mathbf{u}^{n+1}(\mathbf{X}_j^n) \right\}.$$

- Approximation spaces $\mathbf{V}_{i,h}^n$ and $M_{i,h}^n$ now time-dependent.
- Compute $\mathbf{u}_i^{n+1}$ and p_i^{n+1} on $\Omega_{i,\tilde{h}}^n$.

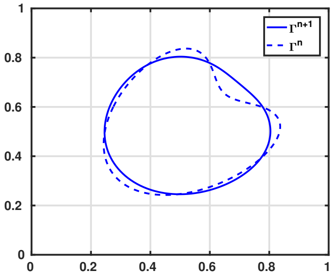
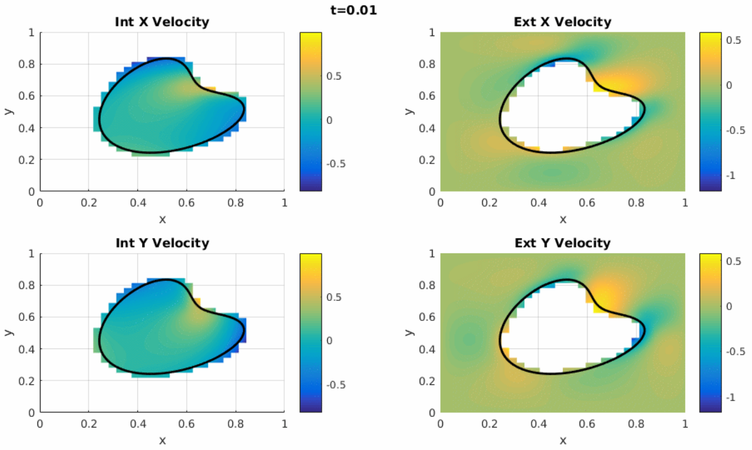


Figure: Subdomains redefined each time step.

Numerical Simulation: Velocity



Cut Finite Element IBM Algorithm

Define:

$\mathbf{Y} = \mathbf{X}^n$	in the Explicit algorithm
$\mathbf{Y} = \mathbf{X}^{n+1}$	in the Semi-implicit algorithm

Steps:

1. Compute the force from jump condition

$$\left([\![(\mu\varepsilon(\mathbf{u}^{n+1}) - p^{n+1})\hat{\mathbf{n}}]\!], \{\mathbf{v}\} \right)_{\Gamma^n} = \kappa \int_0^L \frac{\partial^2 \mathbf{Y}}{\partial s^2} \cdot \{\mathbf{v}\}\, \mathrm{d}s$$

   for $\mathbf{v}_i \in \mathbf{V}^n_{i,h}$, $i = 1, 2$.

2. Solve for $\mathbf{u}_i^{n+1}$ and p_i^{n+1} for $i = 1, 2$ in finite element problem.
3. Update interface location $\mathbf{X}_j^{n+1}$ for $j = 1, \ldots, m$, using

$$\mathbf{X}_j^{n+1} = \mathbf{X}_j^n + \Delta t \{\mathbf{u}^{n+1}(\mathbf{X}_j^n)\} \qquad \text{for } j = 0, \ldots, m.$$

Energy Stability

Define the energy to be the sum of kinetic and elastic potential energy:

$$E^n := \|\mathbf{u}^n\|^2_{L^2(\Omega)} + \kappa \left\| \frac{\partial \mathbf{X}^n}{\partial s} \right\|^2_{L^2([0,L])}.$$

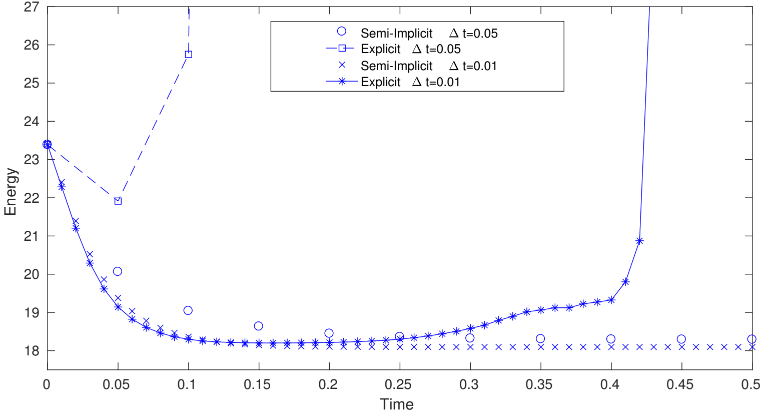
Theorem (D.,Lui, Sarkis, 2018)

Let $\mathbf{u}^{n+1}_{i,h}$, $p^{n+1}_{i,h}$, and $\mathbf{X}^{n+1}_{\tilde{h}}$ be solutions to the finite element problem at time t^{n+1} with $\mathbf{Y} = \mathbf{X}^{n+1}$. Then the following inequality holds:

$$\begin{aligned} E^{n+1} \leq E^n - \Bigg(& \left\|\mathbf{u}^{n+1}_h - \mathbf{u}^n_h\right\|^2_{L^2(\Omega)} + \mu\Delta t \left\|\varepsilon(\mathbf{u}^{n+1}_h)\right\|^2_{L^2(\Omega)} \\ & + \kappa \left\| \frac{\partial \mathbf{X}^{n+1}_{\tilde{h}}}{\partial s} - \frac{\partial \mathbf{X}^{n}_{\tilde{h}}}{\partial s} \right\|^2_{L^2([0,L])} + \frac{2\Delta t}{h} \int_{\Gamma^n_{\tilde{h}} \cup \partial\Omega} [\![\mathbf{u}^{n+1}_h]\!]^2 \, ds \\ & + \Delta t \sum_{i=1}^{2} \left(\mu \mathbf{j}_{i,h}(\mathbf{u}^{n+1}_{i,h}, \mathbf{u}^{n+1}_{i,h}) + 2J_{i,h}(p^{n+1}_{i,h}, p^{n+1}_{i,h}) \right) \Bigg). \end{aligned}$$

Numerical Simulation: Energy

- Parameters: $\kappa = 6$, $\mu = 0.01$, $h = 1/32$, and $\tilde{h} = 1/m$ with $\gamma = 10$.
- Choose m so that $\max_{0 \le j \le m} \left\| \mathbf{X}^0(s_j) - \mathbf{X}^0(s_{j+1}) \right\| < h/2$.



Conclusions

- Improved the finite element immersed boundary method using the cut finite element method.
- Optimal spatial convergence observed in steady-state implementation with Q2-P1 element.
- Unconditional energy stability.
  - Proven in theory.
  - Observed in numerical testing.

Future Work

- Improve temporal convergence of immersed boundary method.
- Prove optimal spatial convergence of CutFEM with immersed boundary using Q2-P1 element.
- Design, analyze, and implement domain decomposition algorithm.
- Cell chemotaxis model and implementation.
    - Couple interior and exterior of cell.

Thank you for your attention!

Questions?

References:

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