Wave scattering, Bayesian estimation of parameters, and detection of signals

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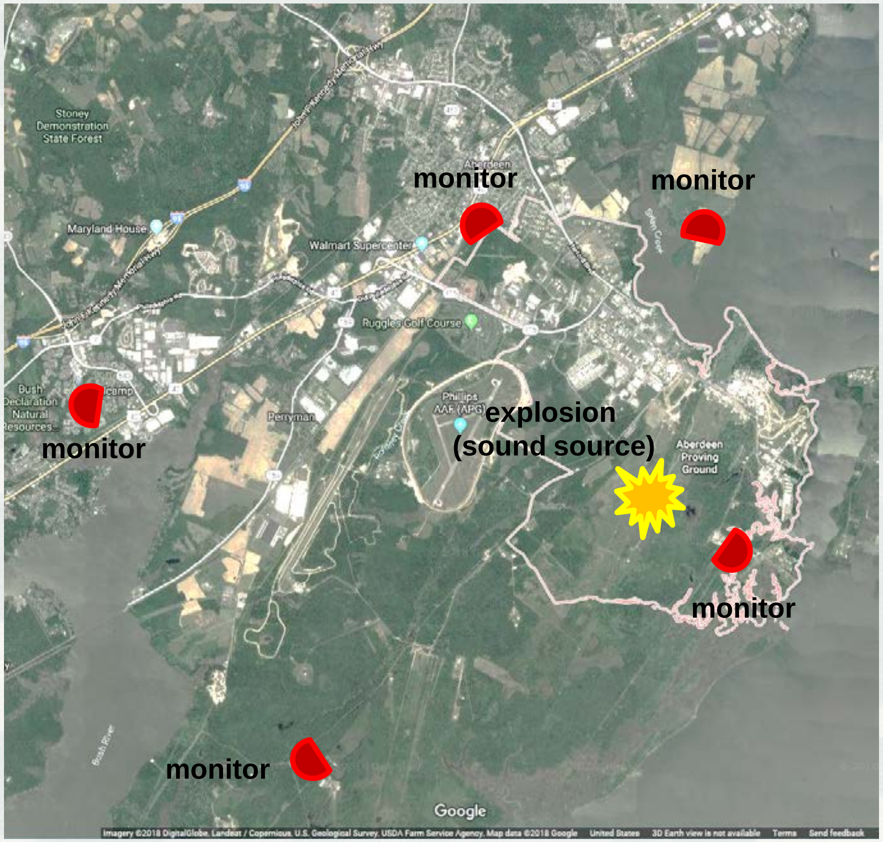
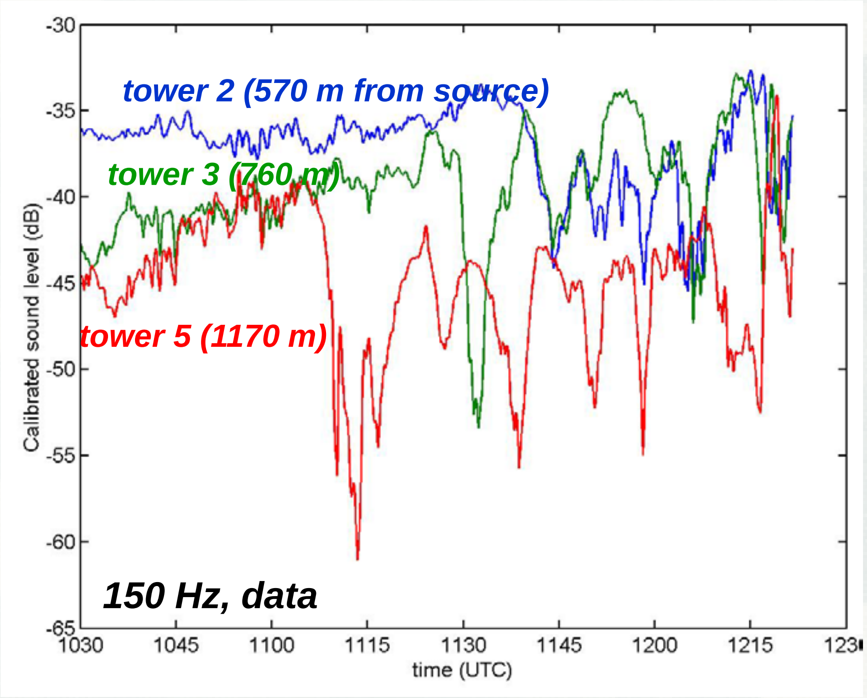
Concept for Adaptive Noise Prediction
(EQ/I RAPIDs Project)

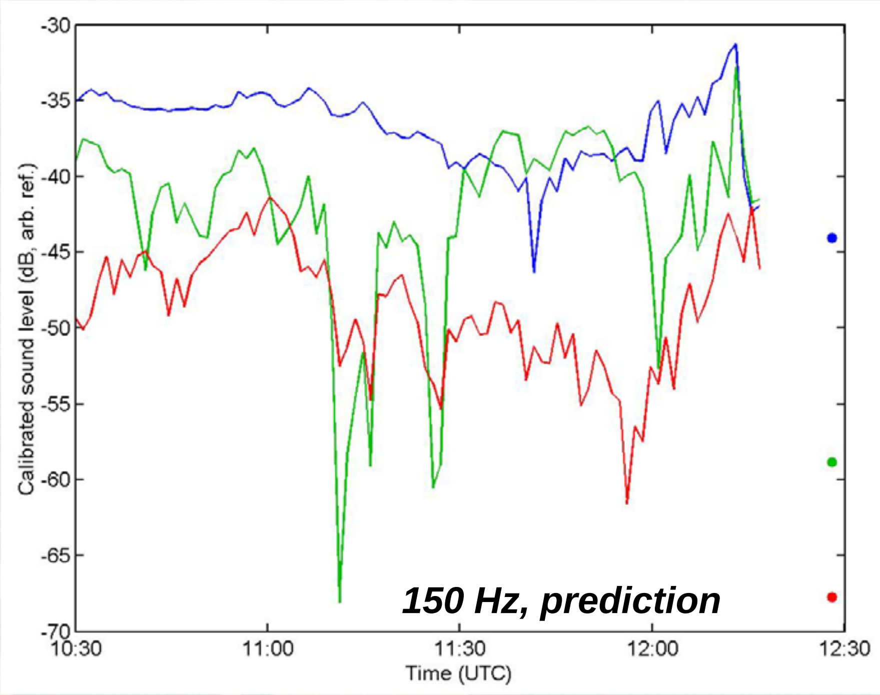


1. We wish to predict the noise levels associated with some activity (e.g., artillery testing at Aberdeen Proving Grounds, MD).
2. We start by collecting information on the noisy activity, terrain, weather forecast, etc., and run our sound propagation model to forecast noise annoyance in adjacent communities.
3. If the noise forecast is favorable, the activity is cleared to proceed.
4. Monitors in the community can provide a real-time check on the accuracy of our forecast.
5. Can we utilize limited monitor data to improve (or *adapt* or *nudge* our forecast) to make it more accurate, and then advise the testing activity whether it can safely continue, or should be curtailed?

But Sounds Levels Random and Difficult to Predict…

Shown below are comparisons between recorded sound levels and predictions from the CASES-99 experiment, which was conducted at night in the Great Plains (Kansas). This experiment provided the best possible scenario for trying to predict sound propagation. (Ref: Wilson et al, *J. Atmos. Sci.* 60, 2473-2486, 2003)



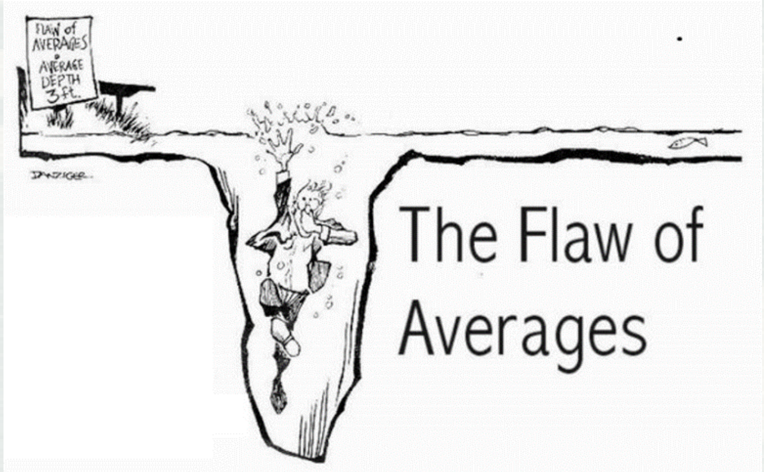


- **Predictions were based on data from a 55-m tower, with wind and temperature sensors every 5 m.**
- **A parabolic equation method was used to predict the sound propagation.**
- **Even with excellent atmospheric data (better than we would normally hope to have), predictive skill for signal variations is very limited.**

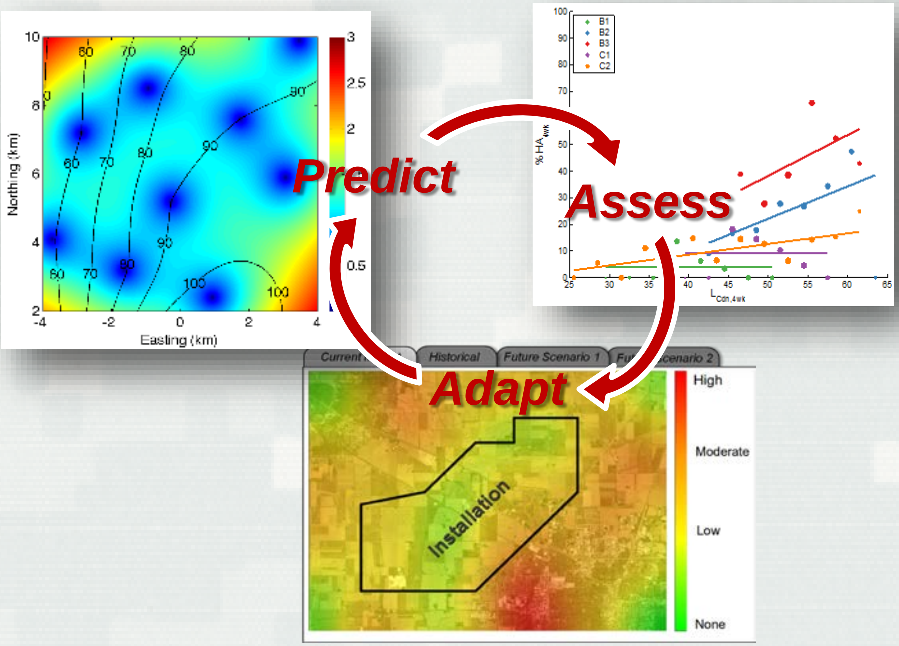
- **What causes the mismatches between predictions and observations at a noise monitor location?**
  - Imperfections in our predictions (due to limited terrain data, finite resolution of atmospheric inputs, limitations of the acoustic propagation model, etc.).
  - Inherent randomness of sound propagation (scattering by atmospheric turbulence, variable ground properties, objects such as vegetation and buildings that can't be resolved, etc.).

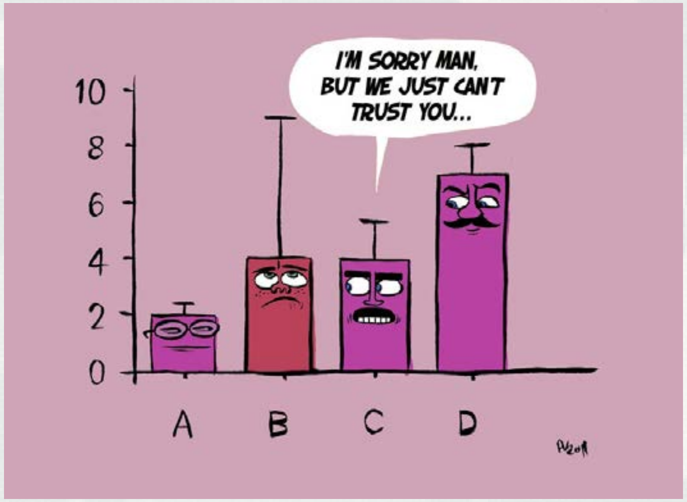
"More decisive? How can I be more decisive?
- I live by the uncertainty principle!"

- **At best, we can predict the *statistical distribution* of the sound level at a monitor (more generally, the signal power at a sensor). The parameters of the distribution are known only imperfectly – they depend on the type of signal, frequency, propagation geometry, intervening terrain, weather conditions, etc.**



Analogous situations may occur in military contexts, e.g., when we predict signal detectability or communications in a complex environment, and then obtain real-time feedback on our predictions as an operation proceeds.

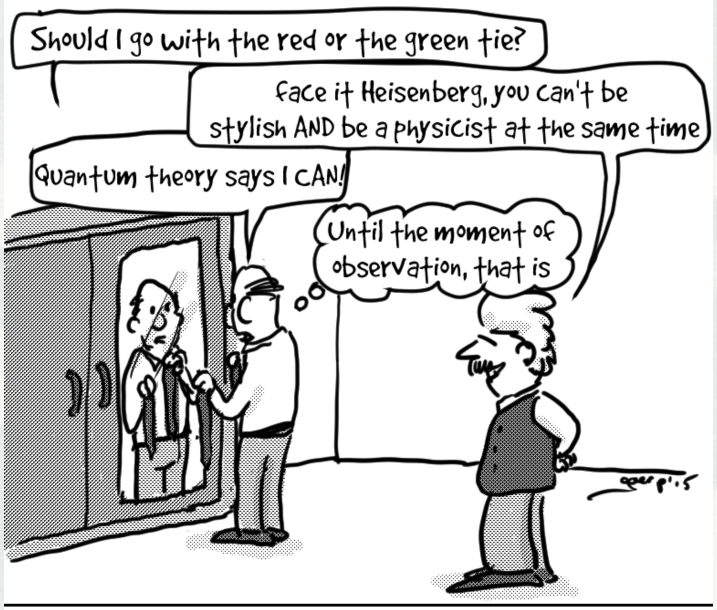




- **Many statistical models, physics-based and empirical, have been formulated for the random signal variations caused by wave scattering.**
- **We will extend such models to include *parametric uncertainties* (uncertainties in the wave scattering parameters).**
- **We also show how modeling parametric uncertainties naturally relates to Bayesian inference of the parameters. This relationship can be exploited to:**
  - Identify statistical models for parametric uncertainties leading to convenient analytical solutions.
  - Develop sequential updating algorithms, which refine an initial prediction of the wave scattering parameters as new signal observations become available.

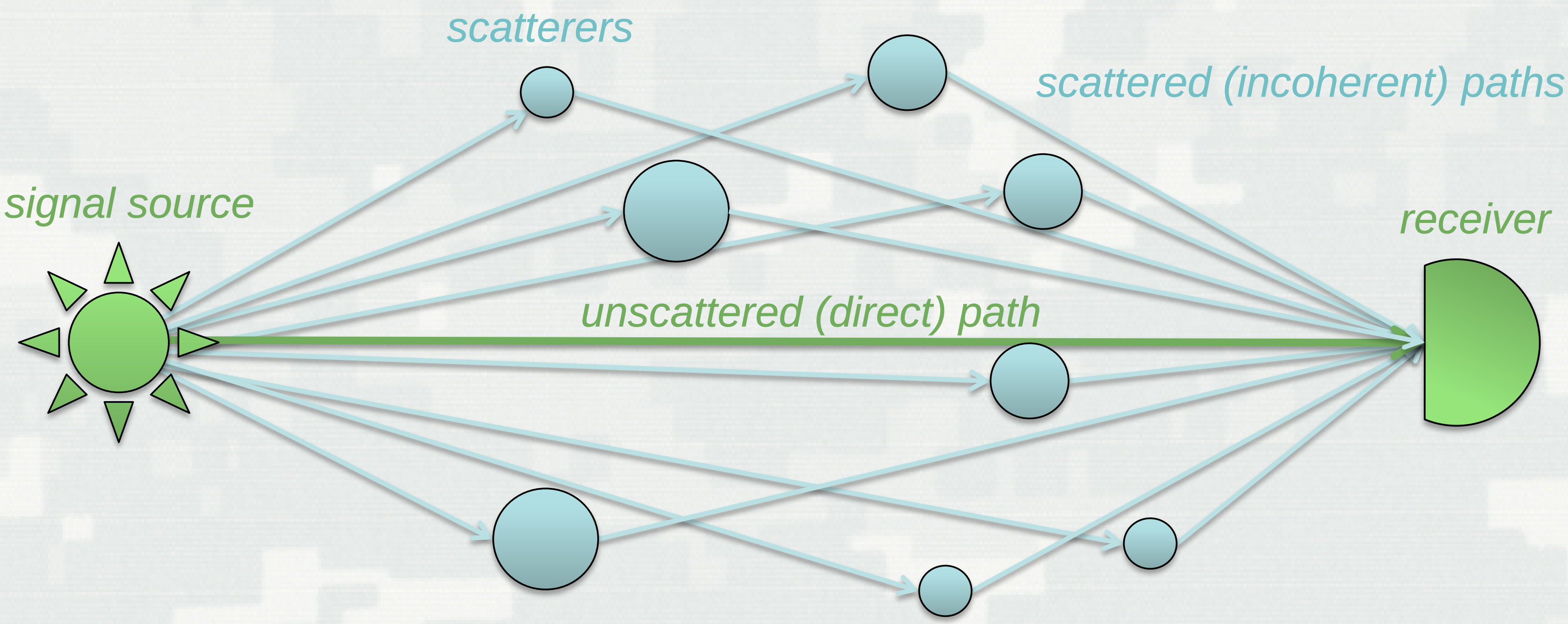
Outline

- **Basic single-variate distributions for scattered signals** (exponential, log-normal, Rician, gamma, generalized gamma)
- **Parametric uncertainties**
  - Compound pdf formulation
  - Turbulent intermittency (exponential/log-normal)
  - K-distribution and its generalization
- **Bayesian methods for incorporating signal observations**
  - Bayes' theorem and relationship to the compound pdf
  - Log-normal/normal (weak scattering)
  - Exponential/inverse gamma (strong scattering)
  - Gamma/inverse gamma (weak or strong scattering)
- **Multi-variate distributions**
  - Log-normal/normal (weak scattering)
  - Wishart DOF 2 (strong scattering)
- **Implications for signal detection**
- **Automated target recognition (ATR) with random signals**
- **Conclusions**



HEISENBERG'S UNCERTAINTY

Simple Conceptual Model



- **The total received signal consists of contributions from an unscattered (direct) path and from multiple randomly scattered (incoherent) paths.**
- ***Weak scattering* means that the direct path dominates; *strong scattering* (*Rayleigh* or *deep fading*) means that the incoherent scattered paths dominate.**
- ***Parametric uncertainty* means that we don't exactly know the statistics of the coherent and/or incoherently scattered waves.**

Distributions for Scattered Signal Power (Notation)

In general, we write the probability density function (pdf) for the scattered signal power as:

signal power

probability density function (pdf) → $p(s|\boldsymbol{\theta})$

parameters of the distribution

Example: For strong scattering, the signal power has an exponential distribution:

$$p(s|m) = \frac{1}{m}\exp\left(-\frac{s}{m}\right) \quad \text{or} \quad p(s|\lambda) = \lambda \exp(-\lambda s)$$

where (first version) $\boldsymbol{\theta} \rightarrow m$ and (second version) $\boldsymbol{\theta} \rightarrow \lambda$. Here $\lambda = 1/m$, and m can be shown to equal the mean power.

(Note: for strong scattering, the signal *amplitude* has a Rayleigh distribution. Throughout this presentation, we will focus on distributions for *power*.)

Other pdfs for Scattered Signal Power

Following are some notable pdfs used for scattered signal power from the literature. (Many more can be found.)

Log-normal (applies to weak scattering, based on the Rytov approximation):

$$p(s|\mu,\phi) = \frac{1}{s\phi\sqrt{2\pi}} \exp\left[-\frac{(\ln s - \mu)^2}{2\phi^2}\right]$$

Rice (weak scattering based on the Born approximation, exact for strong scattering):

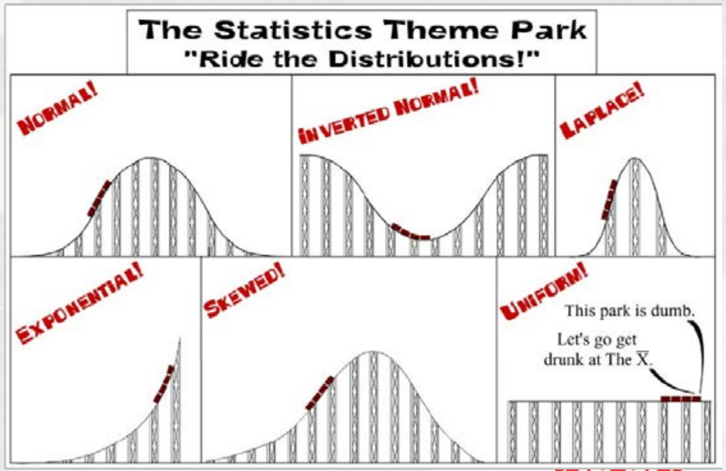
$$p(s|\nu,\varsigma) = \frac{1}{2\varsigma^2} \exp\left(-\frac{s+\nu^2}{2\varsigma^2}\right) I_0\left(\frac{\sqrt{s}\nu}{\varsigma^2}\right)$$

Gamma (weak scattering based on empirical evidence,
exact for strong scattering):

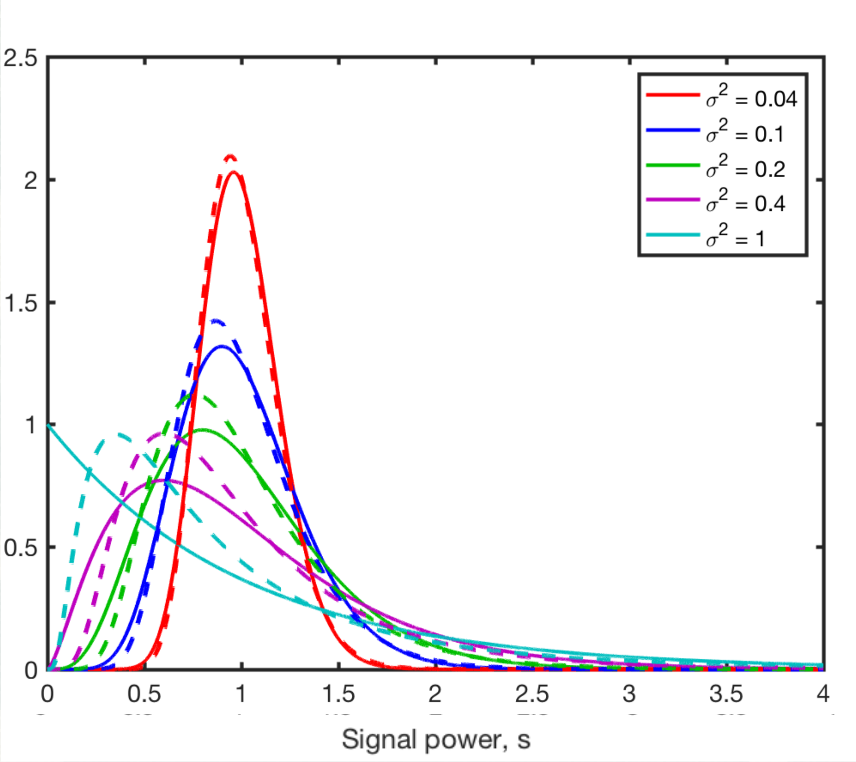
$$p(s|k,\lambda) = \frac{\lambda^k s^{k-1}}{\Gamma(k)} e^{-\lambda s}$$

Generalized gamma (Ewart and Percival 1986)
(reduces to gamma when $b = 1$):

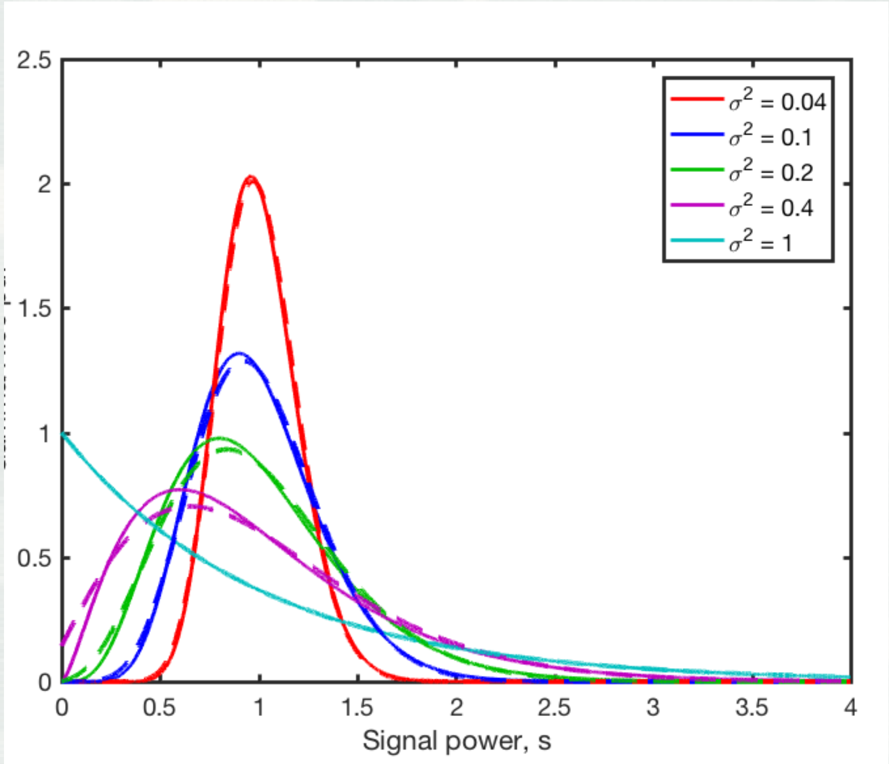
$$p(s|k,\lambda,b) = \frac{b\lambda^{bk} s^{bk-1}}{\Gamma(k)} e^{-(\lambda s)^b}$$



Comparison of Log-normal, Rice, and Gamma pdfs
(with matching means and variances)



Gamma (solid lines) and log-normal (dashed lines) pdfs for various values of the variance normalized by the squared mean. For weak scattering (which is the intended application of the log-normal distribution), the pdfs are nearly identical.

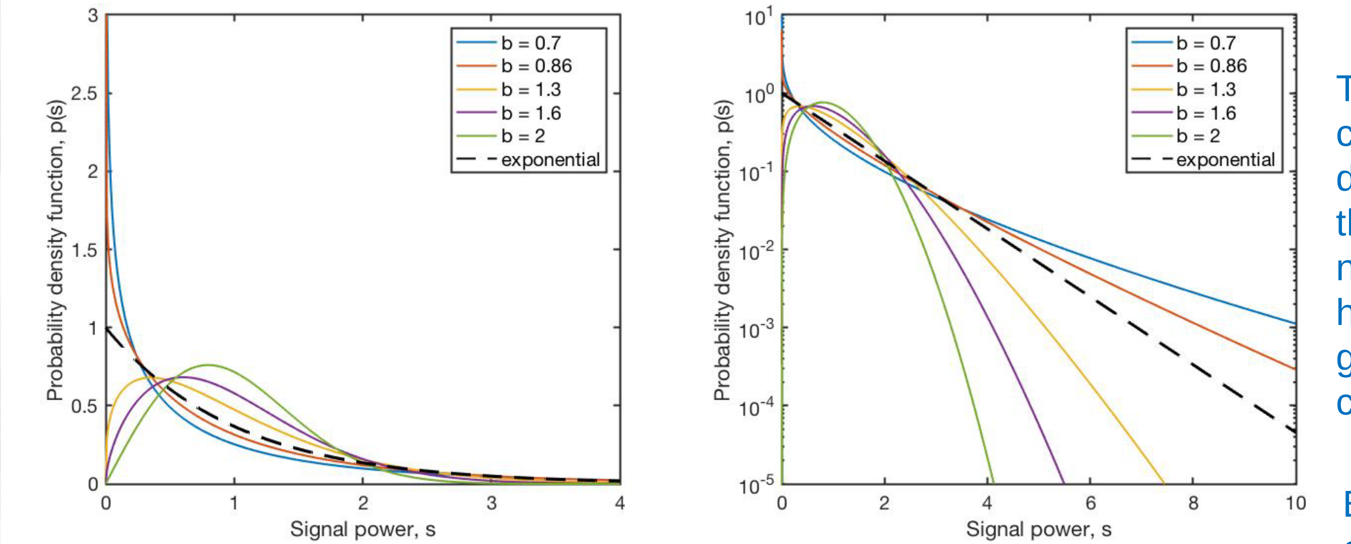


Gamma (solid lines) and Rice (dashed lines) pdfs for various values of the variance normalized by the squared mean. For both strong and weak scattering, the pdfs are nearly identical. At intermediate cases, there are subtle differences.

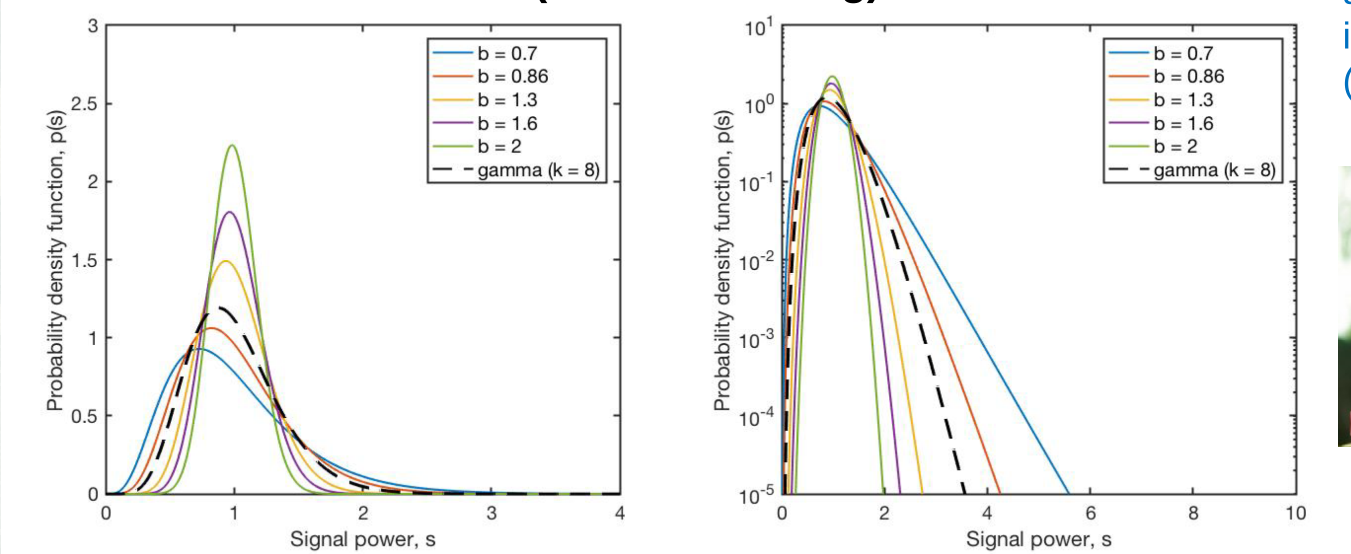
Main point: Log-normal is useful only for weak scattering. Rice and gamma are useful for both weak and strong scattering.

Generalized Gamma Distribution

k = 1 (strong scattering)



k = 8 (weak scattering)



The parameter *b* is seen to control the "tails" of the distribution. As *b decreases*, the pdfs change from a normal-like appearance to having tails exceeding the gamma distribution for the corresponding value of *k*.

Based on empirical fits to ocean acoustic data, Ewing and Percival (1986) find that *b* is *usually* less than 1 (elevated tails are present).

Parametric Uncertainties and the Compound pdf

We use a compound pdf to account for uncertainties:

$$p(s|\boldsymbol{\chi}) = \int \underbrace{p\,(s|\boldsymbol{\theta})}\underbrace{p(\boldsymbol{\theta}|\boldsymbol{\chi})}d\boldsymbol{\theta}$$

pdf describing scattering (depends on parameters θ)

pdf for scattering parameters θ (depends on hyperparameters χ)

Example: Turbulent intermittency with strong scattering
(Gurvich and Kukharets 1986; Wilson et al. 1996)

For strong scattering, the signal power has an exponential pdf:

$$p(s|\boldsymbol{\theta}) = p(s|m) = \frac{1}{m}\exp\left(-\frac{s}{m}\right)$$

By Kolmogorov's refined hypothesis (1962), the structure-function parameters of turbulence (and hence the scattering cross section in the inertial subrange) have a log-normal distribution. Thus

$$p(\boldsymbol{\theta}|\boldsymbol{\chi}) = p(m|\mu,\phi) = \frac{1}{m\phi\sqrt{2\pi}}\exp\left[-\frac{(\ln m - \mu)^2}{2\phi^2}\right]$$

The integral for $p(s|\boldsymbol{\chi}) = p(s|\mu,\phi)$ unfortunately does not have an analytical solution in this case and thus must be determined numerically.

K-Distribution

Andrews and Phillips (2005): "…it has been observed that the lognormal PDF … can underestimate the … tails as compared with measured data. Underestimating the tails of a PDF has important consequences on radar and communication systems where detection and fade probabilities are calculated over the tails of the PDF".

Andrews and Phillips proposed using compound pdfs to provide models with more realistic tails (although they referred to it as a "modulation process".) This led them to the K-distribution.

The K-distribution results from compounding an exponential pdf for *s* (valid for strong scattering) with a gamma pdf (hyperparameters α, β) for the mean power *m*:

$$p(s|\alpha,\beta) = \frac{2\beta}{\Gamma(\alpha)}(\beta s)^{(\alpha-1)/2} K_{\alpha-1}\left(2\sqrt{\beta s}\right)$$

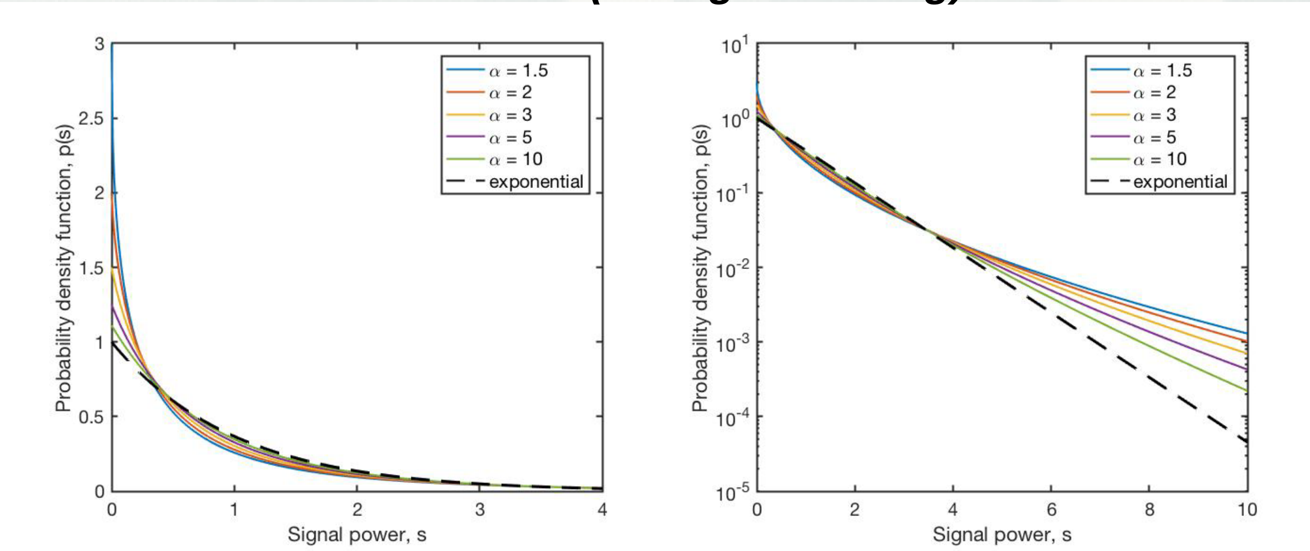
The generalized K-distribution results from compounding a gamma pdf for *s* (with shape parameter *k*; valid for weak or strong scattering) with a gamma pdf (parameters α, β) for the mean power *m*:

$$p(s|\alpha,\beta,k) = \frac{2\beta}{\Gamma(k)\Gamma(\alpha)}(\beta s)^{(k+\alpha-2)/2} K_{\alpha-k}\left(2\sqrt{\beta s}\right)$$

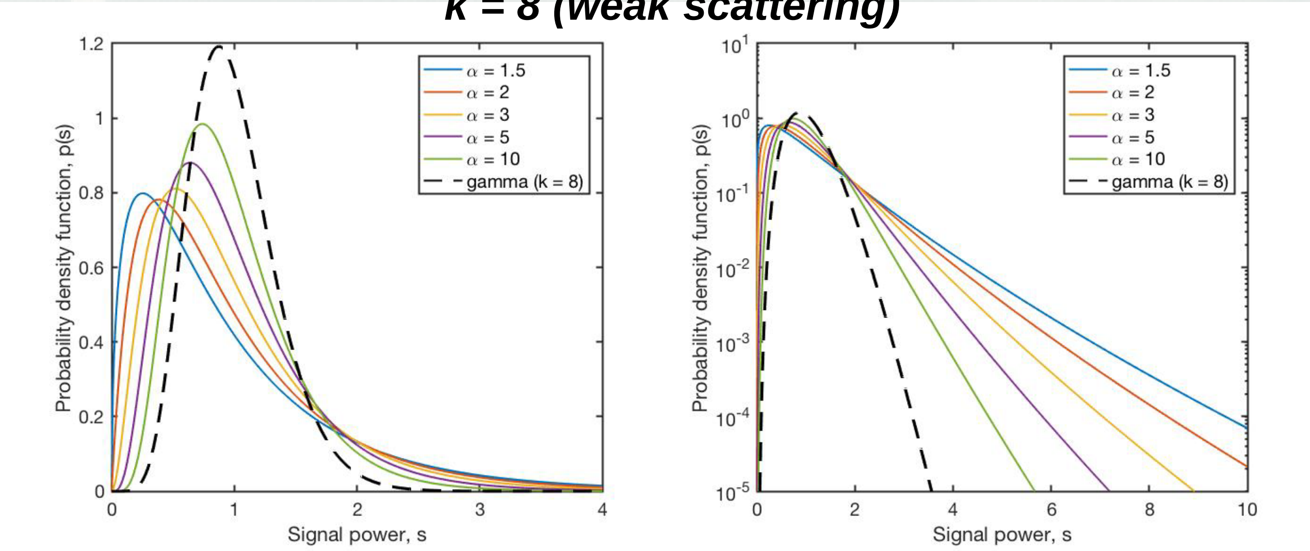
For $k = 1$, the generalized K-distribution reduces to the ordinary K-distribution.

Comparison of Gamma and Generalized K-Distributions

***k* = 1 (strong scattering)**



***k* = 8 (weak scattering)**



The gamma distributions (scattering pdf without any uncertainty) are the black dashed lines. For *k* = 1, the gamma distribution matches the exponential.

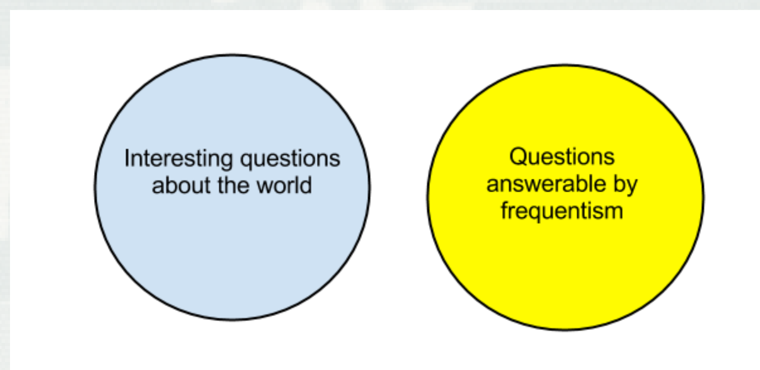
As α increases, the K-distribution converges to the gamma pdf (scattering when no parametric uncertainties are present).

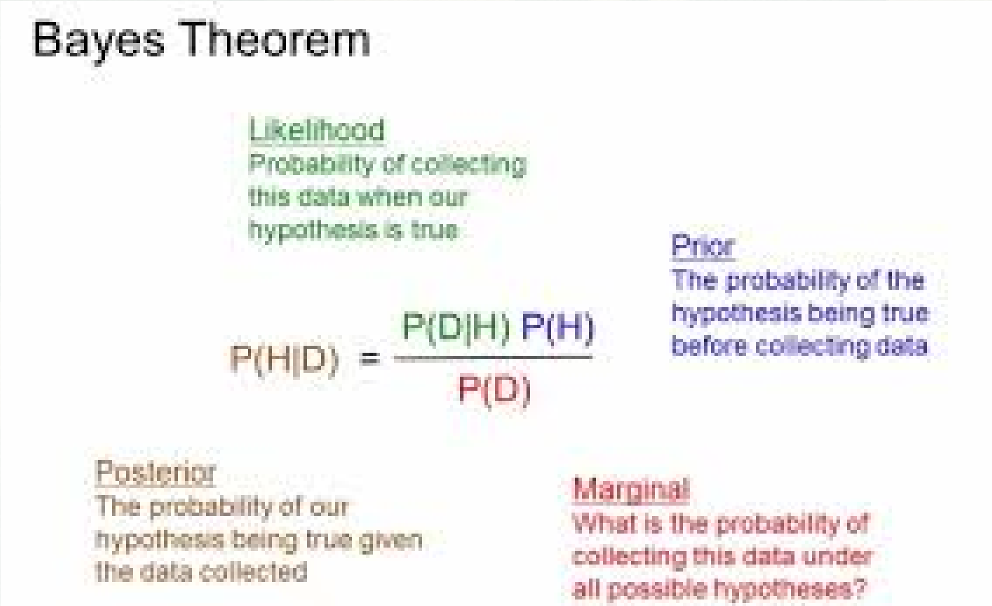
Bayesian Probability: Introduction

- Bayesian probability is based on assigning a prior probability to some hypothesis (belief), and then updating that probability as relevant new data are collected.
- Conventional “frequentist” probability does not involve assignment of prior probabilities; we simply collect data and assess statistics.
- One might think frequentism is superior because it is unbiased by the choice of prior. However, the Bayesian viewpoint has largely triumphed, because it provides a rational way to deal with limited datasets.

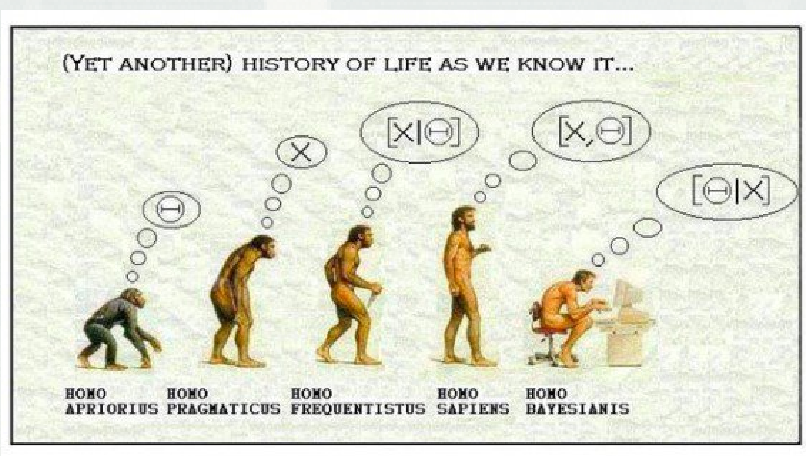
Recommended:

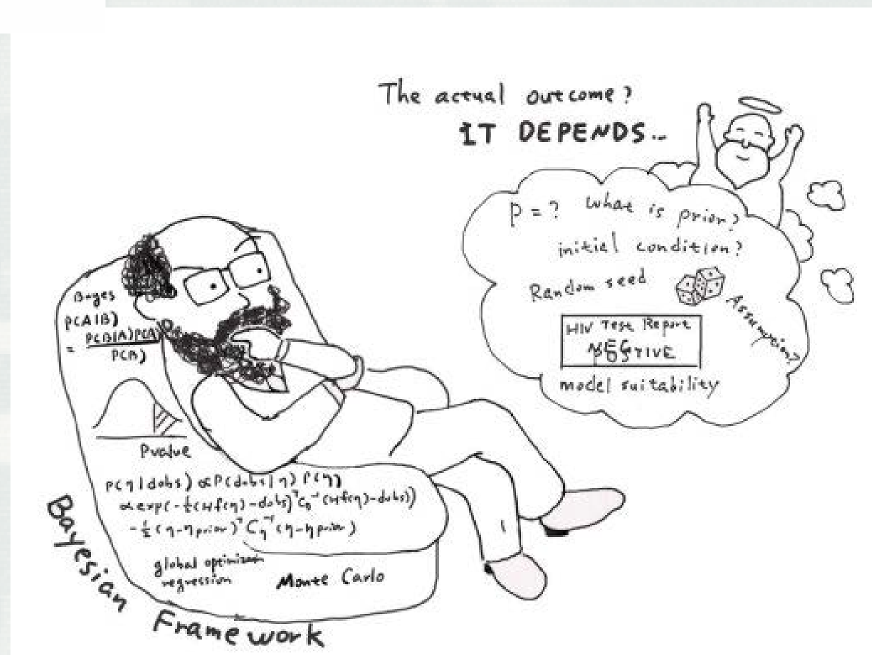
- S. B. McGrayne, *The Theory That Would Not Die: How Bayes’ Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy* (Yale, 2011).
- N. Silver, *The Signal and the Noise: Why So Many Predictions Fail – But Some Don’t* (Penguin, 2012).





Bayes' theorem provides a prescription for updating a *prior* probability as new data become available. The updated probability is called the *posterior*. The posterior at one iteration can be used as the prior at the next.





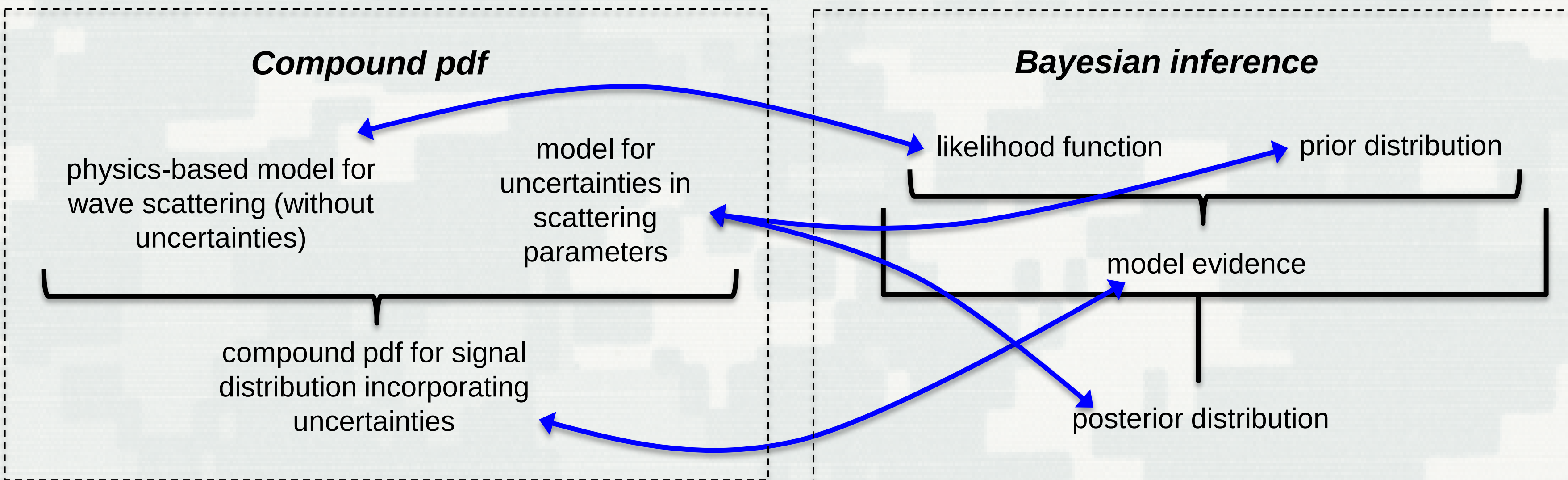
Connection between Bayes' Theorem and the Compound pdf

posterior *likelihood* *prior*

Bayes' theorem:

$$p(\theta|s,\chi) = \frac{p(s|\theta,\chi)\,p(\theta|\chi)}{p(s|\chi)} = \frac{p(s|\theta,\chi)\,p(\theta|\chi)}{\int p\,(s|\theta',\chi)\,p(\theta'|\chi)\,d\theta'}$$

marginal (compound pdf from earlier)



Utilization of Bayesian Conjugate Priors

We are especially interested in cases where the prior and posterior have the same functional form; the prior is then said to be the *conjugate prior* of the likelihood function. This leads to a convenient iterative process where we can sequentially update the hyperparameters as observations of the signal become available.

Strong scattering example: As discussed previously, the signal power has an exponential pdf for strong scattering. In the Bayesian context, this is the likelihood function. The conjugate prior for an exponential likelihood function is known to be the gamma distribution with parameters $\theta \to \{\alpha, \beta\}$. Hence we set

$$p(s|\lambda) = \lambda \exp(-\lambda s) \qquad p(\lambda|\alpha,\beta) = \mathrm{Gamma}(\lambda|\alpha,\beta) = \frac{\beta^{\alpha}\lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda\beta}$$

Integrating, we find for the compound pdf/model evidence:

$$p(s|\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{(s+\beta)^{\alpha+1}}$$

This is called a *Lomax* or *Pareto Type II* distribution. For the posterior, we then find

$$p(\lambda|s,\alpha,\beta) = \frac{(\beta+s)^{\alpha+1}\lambda^{\alpha}}{\Gamma(\alpha+1)} e^{-\lambda(\beta+s)}$$

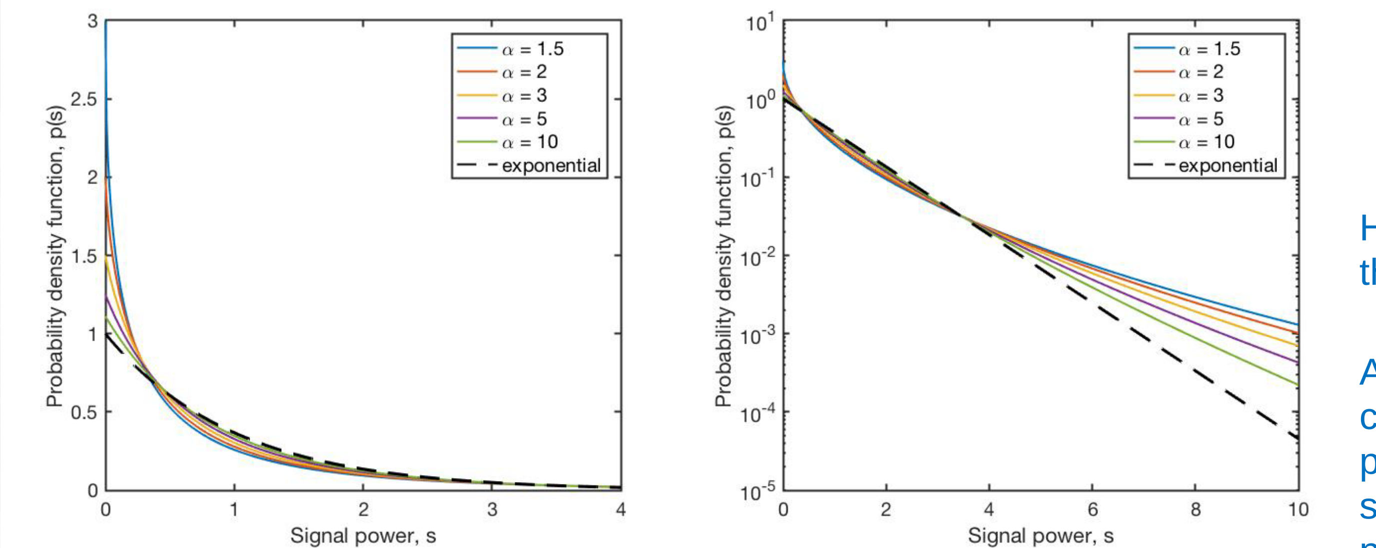
This leads to the following simple formula for updating the distribution of the uncertain parameter λ each time a new signal observation *s* becomes available:

$$p(\lambda|s,\alpha,\beta) = \mathrm{Gamma}(\lambda|\alpha+1,\beta+s) \qquad \alpha \to \alpha+1, \qquad \beta \to \beta+s$$

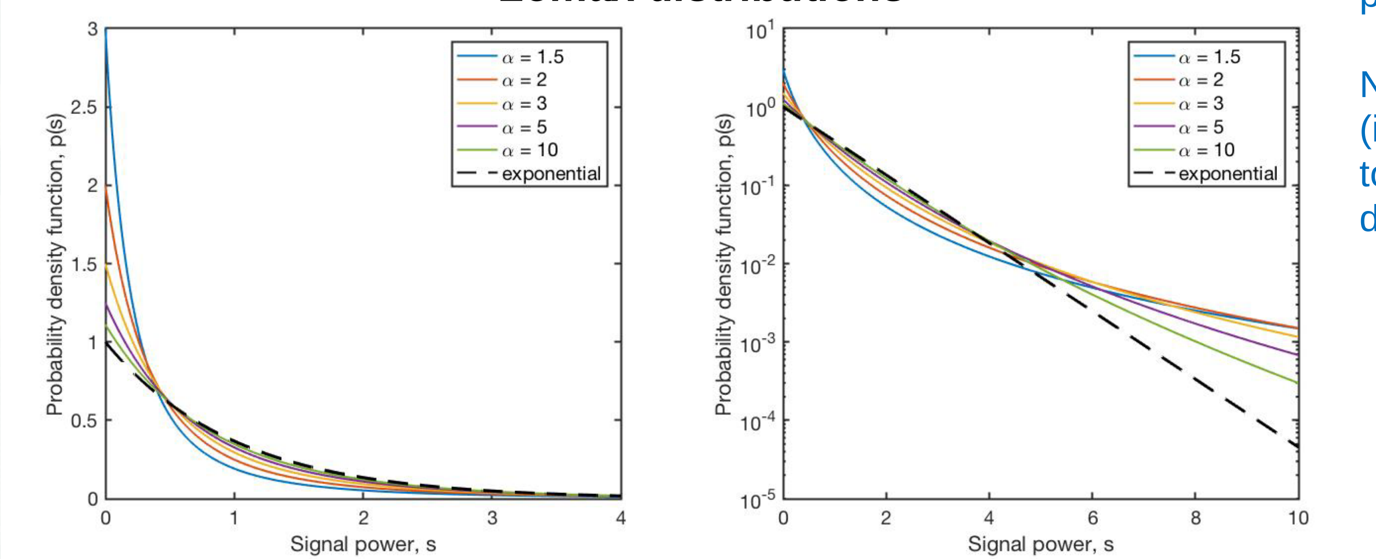
Comparison of K- and Lomax Distributions

(k = 1, strong scattering)

K-distributions



Lomax distributions



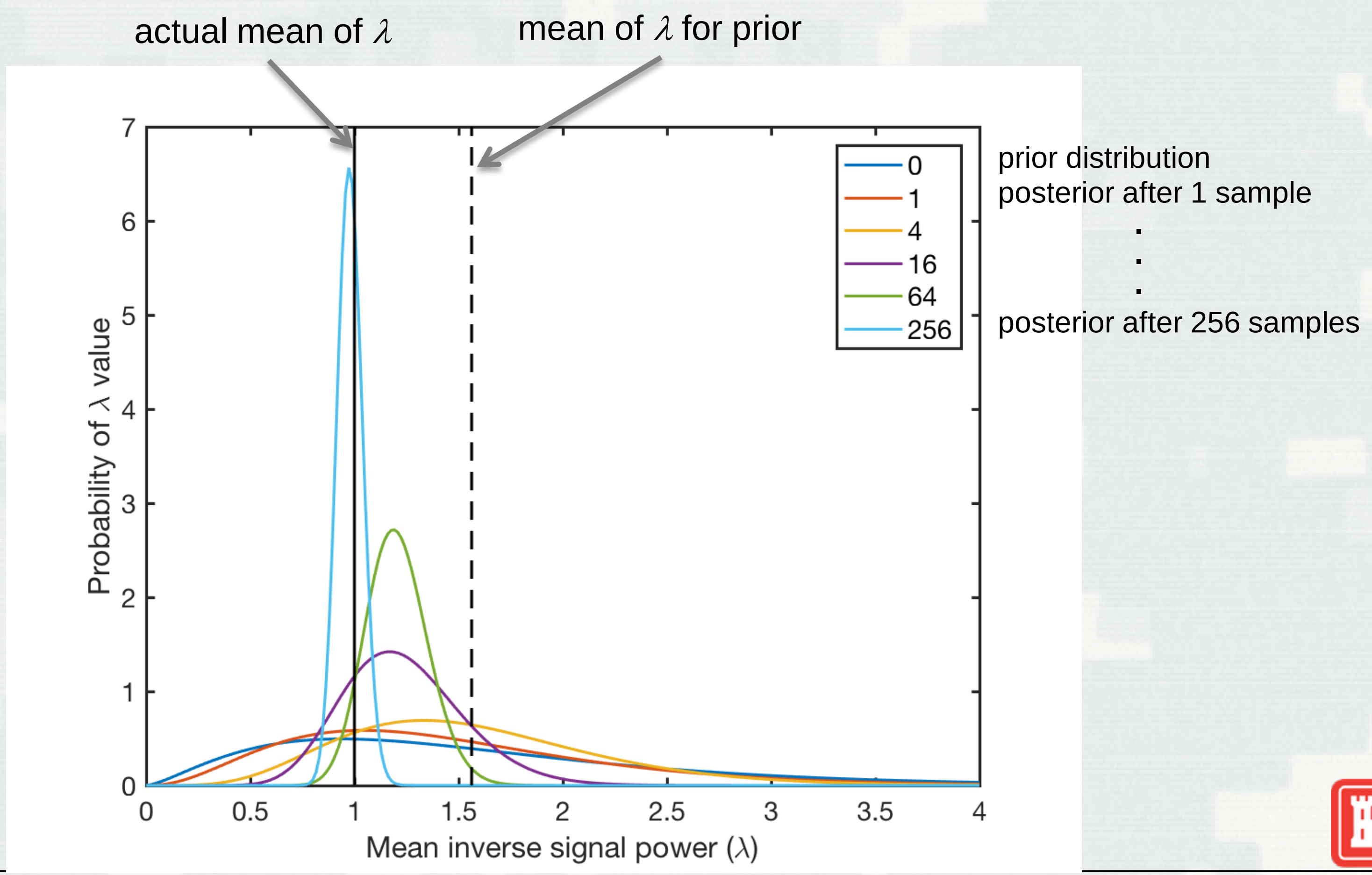
Here, we have set β such that the signal mean equals 1.

As α increases, the pdfs converge to the exponential pdf (that is, to the strong scattering case when no parametric uncertainties are present).

Note that decreasing α (increasing uncertainty) leads to much higher tails in the distribution.

Bayesian Adaptation for Strong Scattering: Simulation

Problem statement: We wish to know the mean received signal power. The signal varies randomly due to strong scattering (exponential pdf). We start with a prior (proposed) distribution for the mean, specifically a gamma pdf, which describes our initially limited knowledge of the mean. We then begin to collect samples of the random signal. After each sample is collected, we can refine the distribution for the mean using Bayes' theorem.



Log-Normal Signal Model with Parametric Uncertainty

(Rytov Approximation for Weak Scattering)

A log-normal pdf can be used to describe a signal with weak scattering. Here we formulate the parametric model for the logarithm of the signal, $\eta = \ln s$:

$$p(\eta|m_\mu, \sigma_\mu^2) = \int p(\eta|\mu) p(\mu|m_\mu, \sigma_\mu^2) d\mu,$$

$$p(\eta|\mu) = \frac{1}{\phi\sqrt{2\pi}} \exp\left[-\frac{(\eta-\mu)^2}{2\phi^2}\right]$$

$$p(\mu|m_\mu, \sigma_\mu^2) = \frac{1}{\sigma_\mu\sqrt{2\pi}} \exp\left[-\frac{(\mu - m_\mu)^2}{2\sigma_\mu^2}\right]$$

We assume that μ (log-mean of the scattered signal strength) is normally distributed and that the variance of μ is known. Performing the integration, we find

$$p(\eta|m_\mu, \sigma_\mu^2) = \frac{1}{\sqrt{2\pi(\sigma_\mu^2 + \phi^2)}} \exp\left[-\frac{(\eta - m_\mu)^2}{2(\sigma_\mu^2 + \phi^2)}\right].$$

Hence the distribution for the log-signal is still normal, although the variance increases.

The Bayesian update for the posterior distribution is:

$$p(\mu|\eta, m_\mu, \sigma_\mu^2) = f_N(\mu|m'_\mu, \sigma'^2_\mu) = \frac{1}{\sigma'_\mu\sqrt{2\pi}} \exp\left[\frac{(\mu - m'_\mu)^2}{2\sigma'^2_\mu}\right]$$

$$m'_\mu = (\sigma_\mu^{-2} + \phi^{-2})^{-1} (\sigma_\mu^{-2} m_\mu + \phi^{-2}\eta) \qquad \sigma'^2_\mu = (\sigma_\mu^{-2} + \phi^{-2})^{-1}$$

Multivariate Log-Normal Distribution

(for Weak Scattering on Multiple Paths)

Extension of the log-normal signal model for the single variate case is straight forward, assuming that the logarithms of the signals follow a multivariate normal distribution:

$$p(\boldsymbol{\eta}|\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \int p(\boldsymbol{\eta}|\boldsymbol{\mu}) p(\boldsymbol{\mu}|\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu)\, d\boldsymbol{\mu},$$

$$p(\boldsymbol{\eta}|\boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^K |\boldsymbol{\Phi}|}} \exp\left[-\frac{1}{2}(\boldsymbol{\eta}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Phi}^{-1} (\boldsymbol{\eta}-\boldsymbol{\mu})\right]$$

$$p(\boldsymbol{\mu}|\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \frac{1}{\sqrt{(2\pi)^K |\boldsymbol{\Sigma}_\mu|}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu}-\mathbf{m}_\mu)^{\mathrm{T}} \boldsymbol{\Sigma}_\mu^{-1} (\boldsymbol{\mu}-\mathbf{m}_\mu)\right]$$

We assume that μ (log-mean of the scattered signal strengths) is normally distributed. The variance of μ is assumed to be known. Performing the integration, we then find

$$p(\boldsymbol{\mu}|\mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \frac{1}{\sqrt{(2\pi)^K |\boldsymbol{\Phi}+\boldsymbol{\Sigma}_\mu|}} \exp\left[-\frac{1}{2}(\boldsymbol{\eta}-\mathbf{m}_\mu)^{\mathrm{T}} (\boldsymbol{\Phi}+\boldsymbol{\Sigma}_\mu)^{-1} (\boldsymbol{\eta}-\mathbf{m}_\mu)\right].$$

The Bayesian update for the posterior distribution is:

$$p(\boldsymbol{\mu}|\boldsymbol{\eta}, \mathbf{m}_\mu, \boldsymbol{\Sigma}_\mu) = \frac{1}{\sqrt{(2\pi)^K |\boldsymbol{\Sigma}'_\mu|}} \exp\left[-\frac{1}{2}(\boldsymbol{\mu}-\mathbf{m}'_\mu)^{\mathrm{T}} (\boldsymbol{\Sigma}'_\mu)^{-1} (\boldsymbol{\mu}-\mathbf{m}'_\mu)\right]$$

$$\mathbf{m}'_\mu = \left(\boldsymbol{\Sigma}_\mu^{-1} + \boldsymbol{\Phi}^{-1}\right)^{-1} \left(\boldsymbol{\Sigma}_\mu^{-1}\mathbf{m}_\mu + \boldsymbol{\Phi}^{-1}\boldsymbol{\eta}\right)$$

$$\boldsymbol{\Sigma}'_\mu = \left(\boldsymbol{\Sigma}_\mu^{-1} + \boldsymbol{\Phi}^{-1}\right)^{-1}$$

Wishart Distribution (n = 2)

(for Strong Scattering on Multiple Paths)

The Wishart distribution is a generalization of the chi-squared distribution to *matrices*. For two degrees of freedom (n = 2), the marginals of the diagonal elements of the matrix have exponential distributions. Thus, the Wishart distribution may an appropriate model for strong scattering along multiple paths.

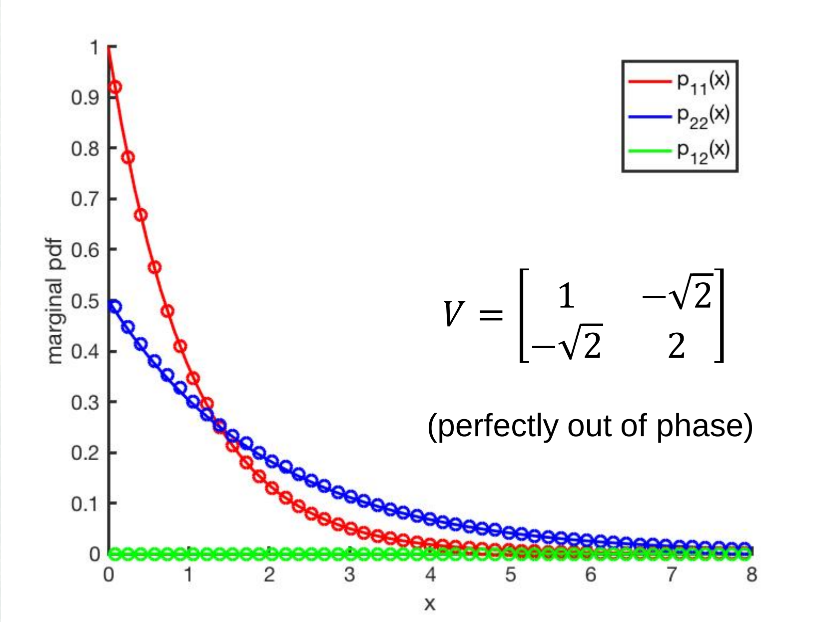
Wishart pdf for $n > p - 1$ degrees of freedom:

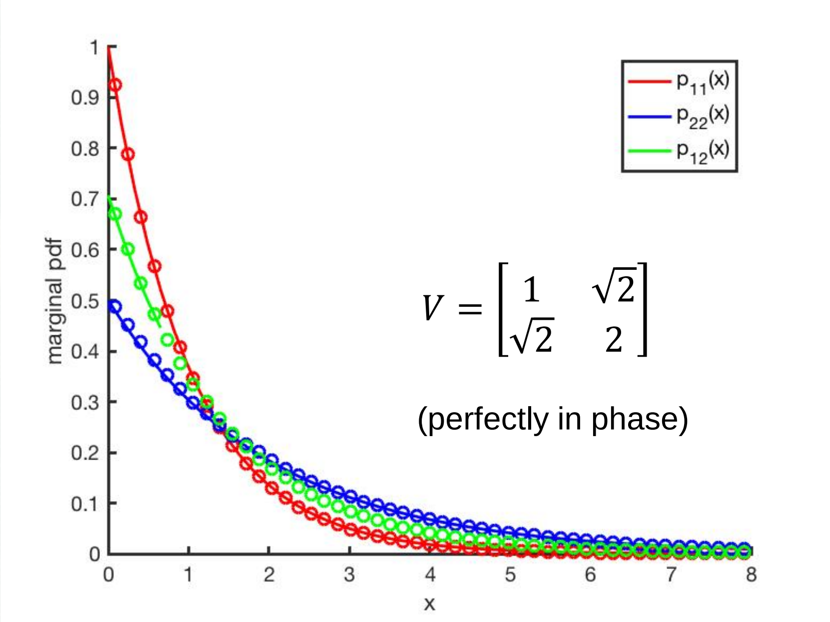
$$p(\mathbf{S}|n, \mathbf{V}) = \frac{|\mathbf{S}|^{(n-p-1)/2}}{2^{np/2}\Gamma_p\left(\frac{n}{2}\right)|\mathbf{V}|^{n/2}} \exp[-\mathrm{tr}(\mathbf{V}^{-1}\mathbf{S})/2]$$

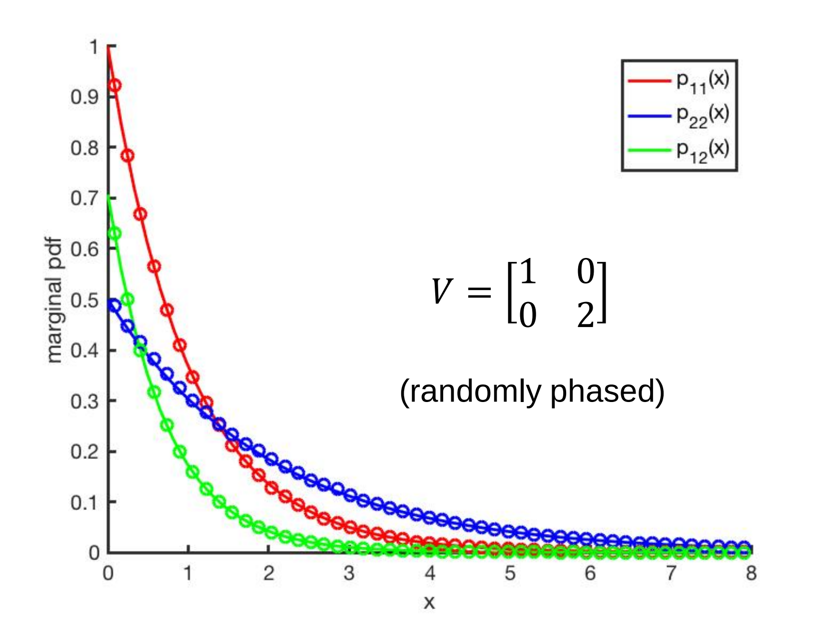
Here, $\mathbf{S}$ is a $p \times p$ symmetric positive definite matrix (containing the signal power at each sensor and their cross correlations), and $\mathbf{V}$ is the scale matrix ($p \times p$ positive definite).

Similarly, the *matrix gamma distribution* generalizes the gamma distribution to positive definite symmetric matrices. It might be useful for both weak and strong scattering.

Wishart Distribution (n = 2)







Shown is a case where the mean on one path is twice the mean on the other.

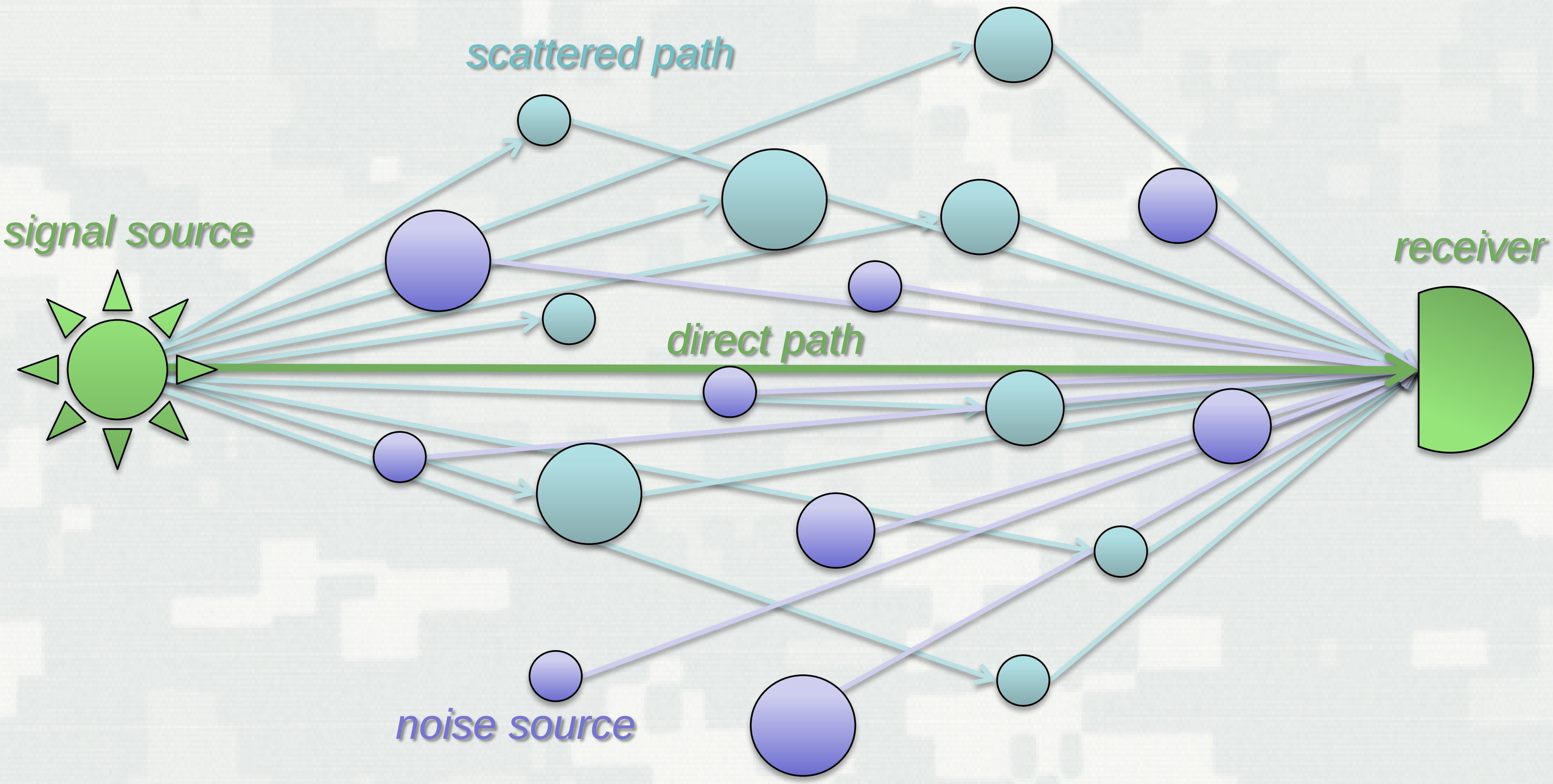
Solid lines are theoretical results and circles are numerical simulations. The marginals p_{11} and p_{22} follow an exponential distribution. The marginal p_{12} has a variance-gamma distribution.

Signal Distributions and Physical Association

Likelihood function (random signal model)	Prior (for mean signal power)	Physical interpretation	Posterior (for mean signal power)	Model evidence (signal model with uncertainty)
Exponential	Gamma (mean)	Strong scattering, single receiver	(non-analytic)	K-distribution
Exponential	Gamma (rate)	Strong scattering, single receiver	Gamma (rate)	Lomax
Exponential	Log-normal	Strong scattering w/turbulent intermittency, single receiver	(non-analytic)	(non-analytic)
Rice	(?)	Weak (Born) or strong scattering, single receiver	(?)	(?)
Gamma	Gamma (mean)	Weak (empirical) or strong, single receiver	(non-analytic)	Generalized K-distribution
Gamma	Gamma (rate)	Weak (empirical) or strong, single receiver	Gamma (rate)	Compound gamma distribution
Log-normal	Normal	Weak (Rytov) scattering, single receiver	Normal	T distribution
Log-normal, multivariate	Multivariate normal	Weak (Rytov) scattering, multiple receivers	Multivariate normal	T distribution, multivariate
Wishart, 2 degree-of-freedom	(matrix inverse gamma?)	Strong scattering, multiple receivers	(matrix inverse gamma?)	(?)
Matrix gamma	(matrix inverse gamma?)	Weak (empirical) or strong, multiple receivers	(matrix inverse gamma?)	(?)

Bayesian conjugate priors are available for the cases shown in red.
Final two rows above are somewhat speculative.

Conceptual Model (with Noise)



The received signal consists of the signal of interest, which propagates along and direct path and multiple randomly scattered paths, and noise from multiple, random sources.

Basic Formulation of Detection Problem

The random scattering and random noise mechanisms lead to a probabilistic distribution for the signal and noise.

Let $p(s, n \mid \theta)$ be the joint probability density function (pdf) for the signal and noise, where θ is the set of parameters for the pdf. Then the probability of false alarm is

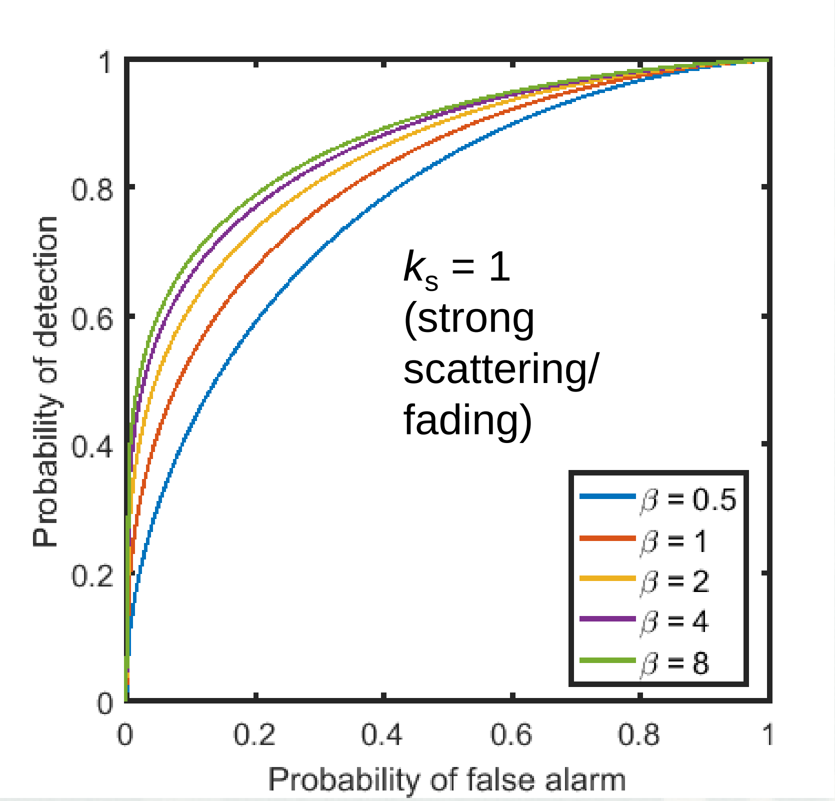
$$P_{\text{fa}}(\theta_n) = \int_{\gamma}^{\infty} p_0\,(x|\theta_n)\,dx \qquad p_0(x|\theta_n) = p(n|\theta_n) = \int p\,(s, n|\theta)\,ds.$$

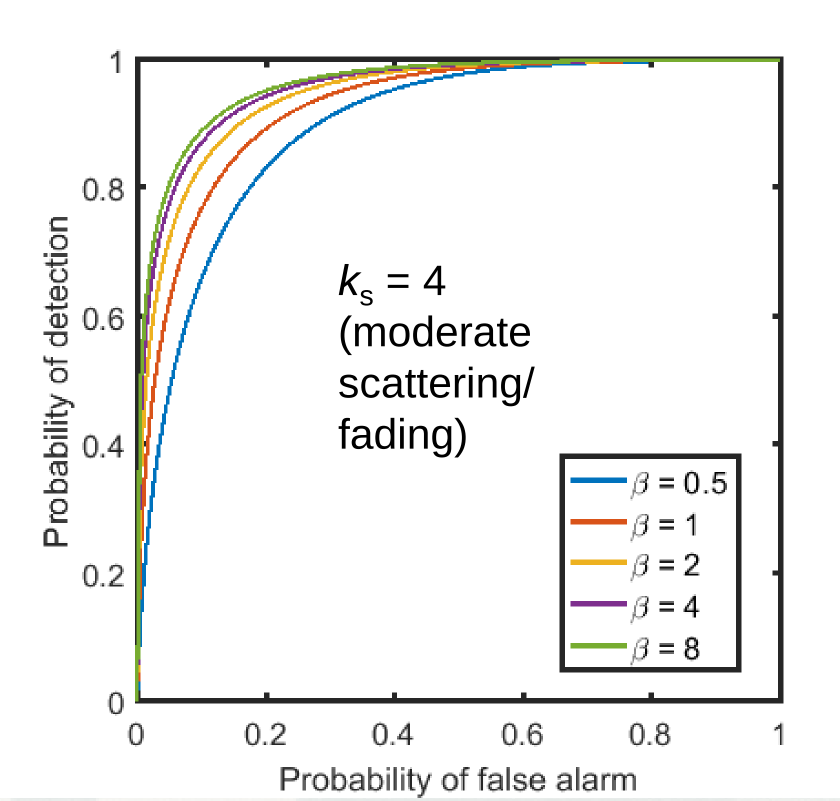
where $p_0(x \mid \theta_{\text{n}})$ is the pdf for the noise alone. The probability of detection is

$$P_{\text{d}}(\theta) = \int_{\gamma}^{\infty} p_1\,(x|\theta)\,dx \qquad p_1(x|\theta) = \int p\,(s, x - s|\theta)\,ds.$$

where $p_1(x \mid \theta)$ is the pdf for the signal plus noise.

ROC Curves with Parametric Uncertainty





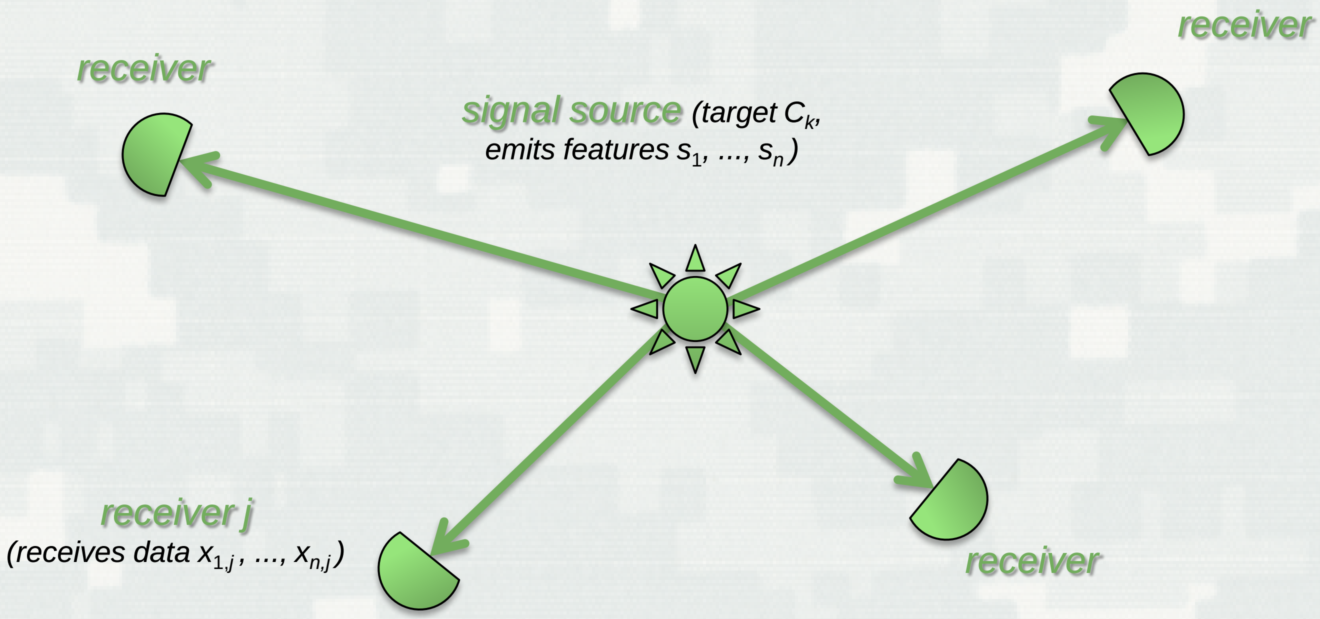
Receiver operating characteristic (ROC) curves corresponding to a gamma-distributed signal with the specified shape factor and gamma-distributed noise with k_n = 4. The signal-to-noise ratio (SNR) is 2, and the hyperparameter β is varied as shown in the legend. (The hyperparameter α is set to $k\beta$+1, so that the mean signal power is 1.)

ATR “Brick Wall”

One of the primary unsolved challenges for performing robust automated target recognition (ATR) is how to compensate the signatures for environmental propagation effects. As a result, ATR algorithms tend to function well only very near the source (where propagation effects are minimized) or in the specific terrain and atmospheric conditions for which they were trained.

- This is particularly true for acoustic and RF signals, which undergo strong frequency-dependent scattering and refraction, particularly in complex environments such as urban, forest, and mountainous. But optical and other types of signals are also impacted.
- In principle, the problem might be attacked by building much larger observational (or simulation) databases than we have now to encompass many additional propagation scenarios. However, we cannot afford to collect data in every terrain condition, ground condition, wind condition, etc., in which a system might operate.

Conceptual Model for ATR with Random Propagation



- **The source signal propagates randomly along multiple transmission paths.**
- **Each sensor receives a version of the feature set (vector) which is distorted during transmission, by a combination of deterministic and random effects.**
- **We model hyperparameters, namely *statistical parameters for the received signal feature set*, at each receiver location.**

Bayesian Classifier w/Propagation Effects

Approach:

1. Use a physics-based model (e.g., an acoustic propagation model supplied with local terrain and weather forecast data) to predict parameters of the pdfs (hyperparameters, $\boldsymbol{\theta}$) for the feature set for each target (C_k), namely $p(\boldsymbol{\theta}|C_k)$.
2. Calculate the probability $p(\mathbf{x}|C_k)$ for an observation of the feature set $\mathbf{x}$ for a given target, using the model from Step 1.
3. Use Bayes' theorem to calculate the posterior probability for each target, $p(C_k|\mathbf{x})$.

Notation

- C_k = target class *k*
- $\mathbf{x}$ = vector of signal features (all features, all receivers)
- $\boldsymbol{\theta}$ = vector of pdf parameters for signal features (all features, all receivers)
- *p*(*a* | *b*) = probability density function for a conditioned on the value of b

prior probability for target C_k

likelihood of observed feature set, for target C_k

probability for observed feature set, given modeled propagation from target C_k

posterior probability for target C_k

$$p(C_k|\mathbf{x}) = \frac{p(C_k)p(\mathbf{x}|C_k)}{p(\mathbf{x})} = \frac{p(C_k)}{p(\mathbf{x})}\int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta}|C_k)d\boldsymbol{\theta}$$

probability for observed feature set (normalizing factor)

probability for modeled signal parameters (hyperparameters), for target C_k

Signal Parameter Updates

One of the strengths of the Bayesian formulation is that it provides a way to update our knowledge of the signal parameters as more observations become available. This helps us overcome the random, unpredictable behavior (scintillations) of acoustic, RF, optical, and other types of signals.

Classifier update:
Provides the probability of a particular target, given the observed feature values.

$$p(C_k|\mathbf{x}) = \frac{p(\mathbf{x}|C_k)p(C_k)}{p(\mathbf{x})}$$

Hyperparameter update:
Updates the propagation prediction for a particular target, given the observed feature values.

$$p(\boldsymbol{\theta}|\mathbf{x}, C_k) = \frac{p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta}|C_k)}{p(\mathbf{x}|C_k)}$$

Sequential updating: The posterior at the current time step becomes the prior at the next time step; that is, $p(C_k|\mathbf{x})$ becomes the new $p(C_k)$, and $p(\boldsymbol{\theta}|\mathbf{x}, C_k)$ becomes the new $p(\boldsymbol{\theta}|C_k)$. **Hence our knowledge of the target probability and the true signal parameters systematically improves as more observations are collected.**

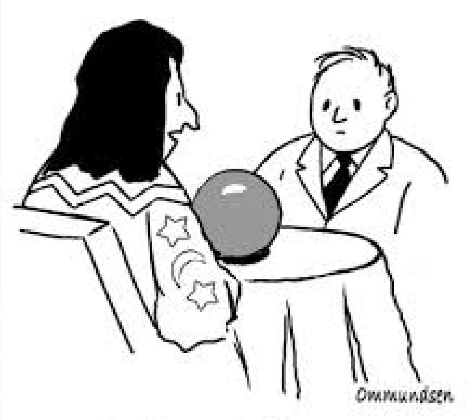
Conclusions

- Connections were examined between physics-based statistical modeling of signals, uncertainties in the wave scattering parameters, and Bayesian inference.

- Uncertainties can be addressed with a compound pdf, which incorporates separate pdfs for the wave scattering process, and for the uncertain wave scattering parameters.

- Uncertainty tends to raise the tails of the signal pdfs, which has important implications for detection and communication system performance.

- In the Bayesian perspective, the scattering models correspond to likelihood functions, which are conveniently paired with their conjugate priors to efficiently update the uncertain signal parameters. The prior distributions, as predicted using an initial forecast based on available weather and terrain data, can then be refined as additional signal observations are collected.



"Is this needed for a Bayesian analysis?"